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ON
A METHOD FOR STUDYING A CERTAIN CLASS OF REGULARITIES IN A SERIES OF OBSERVATIONS WITH APPLICATION TO THE TEMPERATURE-CURVE OF UPPSALA

BY

GUSTAF STRÖMBERG
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Introduction.

In order to find out whether a real influence of the moon could be traced in temperature-variations, I made an investigation [1] as to the existence of a number of terms of periods corresponding to the motion of the sun and the moon in the temperature-curve of Stockholm 1894—1911. I then found small sine-terms of the said periods, but the results were somewhat difficult to interpret. I have thus in the present treatise made an examination of *all* periods between 20 and 40 days in Uppsala temperature, and found that a fully defined trigonometrical series probably exists, representing a part of the temperature-variations. The different terms are found by examining a number of adjacent periods and determining those that correspond to sine-terms with maximal amplitudes.

In the first part of this treatise the theory of the method is explained, and in the second part I have made an application of the method to the temperature-curve of Uppsala 1880—1912.

As this examination of nearly 1000 different periods was a most laborious task I applied in 1913 to Prof. Otto Pettersson of Gothenburg for means to obtain the necessary assistance. Acting on the recommendation of Prof. Pettersson, Dr. Gustaf Ekman and a few other gentlemen in Gothenburg, whose names are unknown to me, put the necessary means at my disposal to accomplish the calculation. I here wish to express my sincere gratitude to those gentlemen for their generous assistance.

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[1] Strömberg: Harmonic analysis of the air-temperature of Stockholm 1894—1911 based on the periods of motion of the sun and the moon. Svenska Hydrografisk-Biologiska Kommissionens Skrifter, Vol 5, Gothenburg 1914.

Theoretical Explanation of the Method.

When studying a series of measured effects in nature, it is in some cases easy for us to find what forces have produced the observed effects and to determine the laws by which the forces work. But this is often possible only in those cases, when the number of acting forces independent of another producing effects of the same order of magnitude is very small and the variations of the supposed forces are well known. When testing whether a supposed force has a perceptible effect on an observed phenomenon, we must employ a statistical method, which can give a conclusion whether a known variation in the supposed force can be traced in the observed effects or not. For instance by means of a simple statistical method we can find that the apparent diurnal and yearly motions of the sun have sensible effects on most meteorological phenomena. Another example is the known fact that a comparison between the sunspot-curve and that curve which represents the variation of the components of terrestrial magnetism shows the existence of a connection between these two phenomena.

In most cases however the variation of the observations is very irregular and we fail to find the laws of the variations. Thus it is of great importance to have a statistical method to find out whether there are any regularities at all in the observed variations. In this treatise I propose to make use of a statistical method for studying a certain class of regularities in a large series of observed effects.

Fourier has shown that a number of observations forming a continuous curve can be represented by a series of sine - (and cosine -) terms with periods equal to the whole, the half, the third part etc. of an original period. We can in this way find a mathematical expression, which represents the observed curve. But this is a method of *interpolation* and cannot be used for *extrapolation.* If we choose different original periods we get different series. Which of these series gives the most probable representation of the future curve?

An analogy from celestial mechanics will in one particular case indicate the most probable trigonometrical series. Provided we have made a large number of observations of the moon and calculated an elliptic orbit of the moon, representing as near as possible the observations, and then formed the differences between observation and calculus, we then get the perturbations in the moon's orbit that arise through the influences of the sun and the planets. If we did not know the law of gravitation or if we could not get mathematical expressions for the perturbations, we could find by a statistical method the different terms in the trigonometrical series representing the perturbations, for instance the "evection", the "variation", the "yearly equation", etc. It is thus necessary to examine a large number of different periods and to determine their effects. In this case the different periods are combinations of the periods in the motion of the moon, the sun and the planets, and we have only to examine a limited number of periods, and it is not necessary to determine the lengths of the periods.

A similar analysis is of practical use in the calculation of the tides. The periods in the motion of the moon and the sun and the combinations of these periods can be traced in the variation of the tides, and we have only to determine the amplitudes and the phase-angles for a small number of different periods, quantities which of course change from one place to another.

In the last two cases we have the great advantage of knowing the acting forces and their periods of variation, and we have thus only to examine a limited number of periods in the observed variations. In those cases, however, when we do not know the forces causing the variations and still less the laws of their action, it is probable that we can employ a similar operation to find a mathematical expression for the observed and the future variations.

Provided we have a long series of observations which practically form a continuous curve we make the following supposition. The mathematical function, which represents the observed curve as a function of the time, has for each moment one and only one value, and its value lies between two finite limits. In the following investigation we shall always make this supposition.

Thus it is possible that a trigonometrical series exists, which represents the said mathematical function, and consequently we have to determine the periods, the amplitudes and the phase-angles of the different sine-terms. For this purpose we must examine all the periods from the shortest to the longest and try to find out those that give the greatest variation. The variation in the different periods is always compared with a sine-line, and we shall thus employ the concepts "periods" and "wave-lengths" as synonyms. From this point of view a prominent character of a curve representing a series of observed effects, may be that sine-terms with constant wave-lengths, amplitudes and phases exist at least for a time, when slowly working forces (terms with long period?) may be considered as constant.

Regularities of this character may thus be calculated and distinguished from real irregularities or regularities of different character. As such a development of a function in a trigonometrical series has a wide application, it is probable that in many cases a large part of the variations in an observed curve can be analysed and determined in this way.

If we have any continuous and one-valued curve, drawn at random, lying between two limits, we can for a certain part of the curve always find a number of sine-terms with arbitrary wave-lengths and with such determined amplitudes and phases that their sum represents the analysed curve. If we in some way determine those wave-lengths whose amplitudes are greatest, the number of terms can be reduced to the smallest number possible. But if we examine another part of the curve, i. e. another curve, we find of course other wave-lengths, that give the greatest variations, that is, we find another development. The developments thus give expressions for interpolation but they cannot be used for extrapolation. Hence, if the theory is correct we must find the same trigonometrical development for different parts of an *observed* curve.

Mean Error of the Amplitudes.

It is of interest to obtain an idea of the mean error of the amplitudes, in order to decide whether the development in a trigonometrical series is a real or only a formal one.

Supposing we have a number of n quantities ("observations") as function of the time (t), the observations may then be denoted by $F(t_n)$, $(n = 1, 2, 3 \ldots n)$. We suppose further that we have already formed the differences between the observed quantities and their average value, so that $\sum\limits_1^n F(t_n) = 0$. We introduce the mean error ε of an observation defined by the equation

$$\varepsilon^2 = \frac{\sum\limits_1^n \left[F(t_n)\right]^2 \cdot {}^{1)}}{n} \tag{1}$$

We then form the difference $F_1(t_n) = F(t_n) - k(\sin gt_n + K)$, and we have to find the values of k, g and K that make this difference a minimum.

The mean error of the quantities $F_1(t_n)$ we may denote by ε_1.

We have then

$$\varepsilon_1{}^2 = \frac{\sum\limits_1^n \left[F_1(t_n)\right]^2}{n}, \tag{2}$$

or

$$n\varepsilon_1{}^2 = \sum_1^n \left[F(t_n)\right]^2 + k^2 \sum_1^n \sin^2(gt_n + K) - 2k \sum_1^n F(t_n) \sin(gt_n + K),$$

thus

$$n\varepsilon_1{}^2 = n\varepsilon^2 + k^2 \sum_1^n \sin^2(gt_n + K) - 2k \sum_1^n F(t_n) \sin(gt_n + K). \tag{3}$$

We may first suppose that we have chosen an arbitrary quantity g $\left(\text{period} = \frac{2\pi}{g}\right)$, and we will find the most probable values of k and K and their mean errors. For this purpose we must determine those values of k and K which will make the right-hand side of equation (3) a minimum.

$$\sum_1^n \sin^2(gt_n + K) = \cos^2 K \sum_1^n \sin^2 gt_n + \sin^2 K \sum_1^n \cos^2 gt_n +$$
$$+ 2 \sin K \cos K \sum_1^n \sin gt_n \cos gt_n =$$
$$= \cos^2 K \left[\frac{n}{2} - \frac{1}{2} \sum_1^n \cos 2gt_n\right] + \sin^2 K \left[\frac{n}{2} + \frac{1}{2} \sum_1^n \cos 2gt_n\right] +$$
$$+ \frac{\sin 2K}{2} \sum_1^n \sin 2gt_n.$$

Thus

$$\sum_1^n \sin^2(gt_n + K) = \frac{n}{2} - \frac{\cos 2K}{2} \sum_1^n \cos 2gt_n + \frac{\sin 2K}{2} \sum_1^n \sin 2gt_n.$$

---

1) For a high value of n it is immaterial whether we have n or n − 1 in the denominator.

But $\Sigma \cos 2gt_n$ and $\Sigma \sin 2gt_n$ are nearly equal to zero when the angles $2gt_n$ are distributed on the whole circumference and vanish exactly when the t_n are uniformly distributed in the period P and the analysed series of observations contains a whole number of the period $\frac{P}{2} = \frac{\pi}{g}$.

We then have, at least approximately,

$$\sum_1^n \sin^2(gt_n + K) = \frac{n}{2}. \tag{4}$$

We have then to find the conditions for a minimum of the function

$$f = \frac{k^2 n}{2} - 2k \sum_1^n F(t_n) \sin(gt_n + K).$$

If we differentiate with respect to k we have the condition

$$kn - 2 \sum_1^n F(t_n) \sin(gt_n + K) = 0$$

or

$$k = \frac{2 \sum_1^n F(t_n) \sin(gt_n + K)}{n}. \tag{5}$$

If we differentiate with respect to K we get

$$\sum_1^n F(t_n) \cos(gt_n + K) = 0. \tag{6}$$

From the two equations (5) and (6)

$$\cos K \sum_1^n F(t_n) \sin gt_n + \sin K \sum_1^n F(t_n) \cos gt_n = \frac{nk}{2}$$

and

$$\cos K \sum_1^n F(t_n) \cos gt_n - \sin K \sum_1^n F(t_e) \sin gt_n = 0,$$

we get

$$\left.\begin{aligned} \frac{nk}{2} \cos K &= \sum_1^n F(t_n) \sin gt_n \\ \frac{nk}{2} \sin K &= \sum_1^n F(t_n) \cos gt_n \end{aligned}\right\} \tag{7}$$

A further condition that k and K in equation (7) may give a minimum of the function f is

$$\frac{\partial^2 f}{\partial k^2} > 0, \quad \frac{\partial^2 f}{\partial k^2} \frac{\partial^2 f}{\partial K^2} - \left[\frac{\partial^2 f}{\partial k \, \partial K}\right]^2 > 0.$$

But

$$\frac{\partial^2 f}{\partial k^2} = n, \quad \frac{\partial^2 f}{\partial K^2} = k^2 n, \quad \frac{\partial^2 f}{\partial k \, \partial K} = 0.$$

The conditions

$$n > 0, \quad k^2 n^2 > 0 \text{ are always fulfilled.}$$

If we put the found values of k and K in equation (3) we get

$$n\varepsilon_1^2 = n\varepsilon^2 + \frac{k^2 n}{2} - k^2 n = n\varepsilon^2 - \frac{k^2 n}{2}$$

or

$$\varepsilon_1^2 = \varepsilon^2 - \frac{k^2}{2}. \tag{8}$$

By using the method of least squares instead of the above investigation, we should of course have found the same equations for determining k and K.

The equations of condition would then have been

$$k \cos K \sin gt_n + k \sin K \cos gt_n = F(t_n),$$

and the normal equation would be

$$k \cos K \sum_1^n \sin^2 gt_n + k \sin K \sum_1^n \sin gt_n \cos gt_n = \sum_1^n F(t_n) \sin gt_n$$

$$k \cos K \sum_1^n \sin gt_n \cos gt_n + k \sin K \sum_1^n \cos^2 gt_n = \sum_1^n F(t_n) \cos gt_n.$$

But

$$\sum_1^n \sin^2 gt_a = \sum_1^n \cos^2 gt_n = \frac{n}{2}$$

and

$$\sum_1^n \sin gt \cos gt = \frac{1}{2} \sum_1^n \sin 2gt_n = 0.$$

Hence

$$\left.\begin{aligned} \frac{nk}{2} \cos K &= \sum_1^n F(t_n) \sin gt_n \\ \frac{nk}{2} \sin K &= \sum_1^n F(t_n) \cos gt_n \end{aligned}\right\} \tag{7}$$

From equations (7) we see that the weight of the quantity k is $\frac{n}{2}$.

Let ε_k be the mean error of k.

Therefore $$\varepsilon_k^2 = \frac{2\varepsilon_1^2}{n} = \frac{2\varepsilon^2 - k^2}{n}. \tag{9}$$

From this formula we find that a large value of k will give a small value of ε_k, i. e. the probability of the existence of the term $k \sin(gt + K)$ is greatest when k has its largest value. We must then determine g in such a way, that the amplitude k obtains its largest value. In this investigation this condition is of fundamental importance.

Equation (9) makes it possible to decide if a given series of observations probably contains constant sine-terms. If a curve has no regularities at all or regularities of different character, the values of k found by the analysis must be nearly equal to ε_κ. As we shall see at the application of the method, the mean error in the difference between the observed temperature in Uppsala and the normal temperature for the season is $\pm 3^0.3$ C., when the observed temperature-curve is smoothed out by taking 5 days' means for every day. If no constant harmonics exist the found value of the amplitudes would be approximately

$$k = \varepsilon_k = \sqrt{\frac{2\varepsilon^2}{n}} = \sqrt{\frac{2}{n}}\, 3^0.3 .$$

At the analysis for the time 1880—1912 we have $n = 4800$ and k or ε_k would be

$$\pm 0^0.067 .$$

But the analysis gives some harmonics with amplitudes $k > 0^0.30$ (see Fig. 3), and it is thus probable that such harmonics are real ones.

Elimination of Variations of Short Periods.

When we want to examine all the wave-lengths in a series of observations at many investigations it is convenient to separate terms of short periods from those of longer periods, which we can do by forming a smooth curve.

If for instance at a series of daily observations we compute 10 days' means for every day, an eventual term with a period shorter than 10 days will be almost eliminated, but this operation has no influence on a term with a period of one year. Such a calculation of means of a few days is often convenient as we can then reduce the number of equations of condition.

Thus it is necessary to have a general method of calculating the influence of this formation of means on the different periods. We will in the following suppose the independent variable to be the time and the unit to be one day, but it is easy to apply the formulæ to other variables and units.

If we suppose that the observations can be represented by a sum of sine-terms with different periods, it is important to know the "factor of reduction" which the formation of means will involve in the different amplitudes. If we consider the observations as a series of isolated points, it is convenient to form means of an odd number of observations. The calculated mean must be valid for the mean of the corresponding values of the time. For every sine-term we have then the following formula for the mean of n observations valid for the mean (t_0) of the n equidistant moments:

$$\frac{1}{n}\sum_{m=-\frac{n-1}{2}}^{\frac{n-1}{2}} k\sin(gt+K+mg) = \frac{k}{n}\sin(gt+K)\sum_{m=-\frac{n-1}{2}}^{\frac{n-1}{2}}\cos mg +$$

$$+\frac{k}{n}\cos(gt+K)\sum_{m=-\frac{n-1}{2}}^{\frac{n-1}{2}}\sin mg.$$

But

$$\sum_{m=-\frac{n-1}{2}}^{\frac{n-1}{2}}\cos mg = \frac{\sin\frac{ng}{2}}{\sin\frac{g}{2}}$$

and

$$\sum_{m=-\frac{n-1}{2}}^{\frac{n-1}{2}}\sin mg = 0.$$

We have then

$$\frac{1}{n}\sum_{m=-\frac{n-1}{2}}^{\frac{n-1}{2}} k\sin(gt+K+mg) = \frac{k\sin\frac{ng}{2}}{n\sin\frac{g}{2}}\sin(gt+K).$$

The factor of reduction of the amplitude will then be

$$R = \frac{\sin\frac{ng}{2}}{n\sin\frac{g}{2}}. \tag{10}$$

The period is $P = \frac{2\pi}{g}$.

Thus
$$R = \frac{\sin\frac{n\pi}{P}}{n\sin\frac{\pi}{P}}. \tag{11}$$

If we calculate for instance the yearly variation (considered as a sine-term) in the temperature from a series of observations, each of which represents 30 days' mean temperature, we have

$$\frac{n}{P} = \frac{30}{365} = 0.822$$

and
$$R = \frac{\sin 14^\circ.80}{30\sin 0^\circ.4932} = 0.989.$$

If the observations are not isolated points but form a continuous curve, we can find a more accurate and simple form for R. We suppose then that we have computed means of n days *for every moment.* The smooth curve of n days' means is then fully defined and we have to calculate the factors of reduction for periods of different lengths.

The term $k\sin(gt + K)$ is then transformed into

$$Rk\sin(gt+K) = \frac{1}{ng}\int_{-\frac{ng}{2}}^{+\frac{ng}{2}} k\sin(gt+K+v)\,dv =$$
$$= \frac{2k}{ng}\sin(gt+K)\sin\frac{ng}{2}.$$

Thus
$$R = \frac{\sin\frac{ng}{2}}{\frac{ng}{2}} = \frac{\sin\frac{n\pi}{P}}{\frac{n\pi}{P}}. \tag{12}$$

For small values of $\frac{\pi}{P}$, as in the last example, the two formulæ (11) and (12) give the same results. Formula (12) must be employed when the observations themselves are the means of several original ones. As it is of interest to have a table of the values of R, we here give such a table calculated on the basis of formula

(12); this table we shall make use of at the application. When $\frac{n}{P}$ is an integral number the value of R is zero. [1])

Table 1.

Table of the factor of reduction R.

$\frac{n}{P}$	R	$\frac{n}{P}$	R	$\frac{n}{P}$	R
0	+ 1.000	1.2	— 0.156	2.6	+ 0.117
0.02	+ 1.000	1.3	— 0.198	2.7	+ 0.095
0.04	+ 0.997	1.4	— 0.216	2.8	+ 0.067
0.06	+ 0.994	1.43	— 0.218 min	2.9	+ 0.034
0.08	+ 0.990	1.5	— 0.212	3.0	0.000
0.10	+ 0.983	1.6	— 0.189	3.47	— 0.091 min
0.2	+ 0.936	1.7	— 0.151	4.0	0.000
0.3	+ 0.858	1.8	— 0.104	4.47	+ 0.071 max
0.4	+ 0.757	1.9	— 0.052	5.0	0.000
0.5	+ 0.637	2.0	0.000	5.48	— 0.058 min
0.6	+ 0.505	2.1	+ 0.047	6	0.000
0.7	+ 0.368	2.2	+ 0.085	6.48	+ 0.049 max
0.8	+ 0.234	2.3	+ 0.112	7	0.000
0.9	+ 0.109	2.4	+ 0.126	7.49	— 0.042 min
1.0	0.000	2.46	+ 0.128 max	8	0.000
1.1	— 0.089	2.5	+ 0.127		

If we thus have a sum of sine-terms and for instance make a continuous curve representing means of 10 days for every moment, a term with the period 6.67 days $\left(\frac{n}{P} = \frac{10}{6.67} = 1.5\right)$ has its amplitude reduced by the factor $R = 0.212$ and the sign of the term is changed.

It is to be observed that such a graduation of a sum of sine-terms cannot change the phases. If we suppose the amplitudes to be positive, the phase-angles may nevertheless differ by 180°.

---

[1]) The maxima and minima of R are determined by the equation

$$\tan \frac{n\pi}{P} = \frac{n\pi}{P} \left[\frac{n}{P} = 0, \ 1.4803, \ 2.4590, \ 3.4709, \ 4.4747 \ \text{etc.}\right]$$

On the Mutual Influences of the Terms.

Compound Waves.

Equation (9) is only correct in the case when the determinations of the different terms can be made in such a way, that the terms are independent of each other, i. e. if we choose another value for k and K for a term, this cannot influence the determination of another term. For such separate determinations of the terms it is necessary to have a large number of observations, and we may then form a number of average values each of which is found by adding those observations which correspond to equidistant values of the time, where the constant difference between two such argument is equal to one period of the term that is to be determined. We shall now try to find the influences of the different terms upon each other.

Let us suppose that the function F(t) can be expressed by the equation

$$F(t) = \sum_{0}^{n} k_n \sin (g_n t + K_n) = k \sin (gt + K) + \sum_{1}^{n} k_n \sin (g_n t + K_n).$$

We suppose further that our series of observations contains an integral number $N_2 - N_1$ of the period $P = \frac{2\pi}{g}$, where N_1 is the number of periods P to the beginning and N_2 to the end of the time for which the analysis is carried out. We have to determine a term $k \sin (gt + K)$ of the period $P = \frac{2\pi}{g}$.

We then form the means

$$\frac{1}{N_2 - N_1} \sum_{N_1}^{N_2 - 1} F(t + NP) = k \sin (gt + K) +$$

$$+ \frac{1}{N_2 - N_1} \sum_{1}^{n} \sum_{N_1}^{N_2 - 1} k_n \sin \left(g_n t + K_n + \frac{g_n}{g} 2\pi N \right).$$

In the determination of the term $k \sin(gt + K)$ the perturbations of other terms are defined by the expression

$$\frac{1}{N_2 - N_1} \sum_{1}^{n} \sum_{N_1}^{N_2-1} k_n \sin\Big((g_n t + K_n + \frac{g_n}{g} 2\pi N\Big) \cdot$$

Let

$$\left.\begin{aligned} \frac{g_n}{g} &= \frac{P}{P_n} = 1 + \varepsilon_n \\ \text{or} \qquad \varepsilon_n &= \frac{P - P_n}{P_n} \end{aligned}\right\} \qquad (13)$$

Thus

$$\sum_{N_1}^{N_2-1} \sin\left(g_n t + K_n + \frac{g_n}{g} 2\pi N\right) =$$

$$= \sin(g_n t + K_n) \sum_{N_1}^{N_2-1} \cos 2\pi N \varepsilon_n + \cos(g_n t + K_n) \sum_{N_1}^{N_2-1} \sin 2\pi N \varepsilon_n .$$

But

$$\sum_{1}^{m} \sin ma = \frac{\cos\frac{a}{2} - \cos\frac{2m+1}{2} a}{2 \sin\frac{a}{2}}$$

or

$$\sum_{N_1}^{N_2-1} \sin ma = \sum_{1}^{N_2-1} \sin ma - \sum_{1}^{N_1-1} \sin ma =$$

$$= \frac{\cos\frac{2N_1-1}{2} a - \cos\frac{2N_2-1}{2} a}{2 \sin\frac{a}{2}} = \frac{\sin\frac{N_2-N_1}{2} a \ \sin\frac{N_1+N_2-1}{2} a}{\sin\frac{a}{2}} .$$

Further we have

$$\sum_{1}^{m} \cos ma = \frac{\sin\frac{2m+1}{2} a - \sin\frac{a}{2}}{2 \sin\frac{a}{2}}$$

or

$$\sum_{N_1}^{N_2-1} \cos ma = \sum_{1}^{N_2-1} \cos ma - \sum_{1}^{N_1-1} \cos ma =$$

$$= \frac{\sin\frac{2N_2-1}{2} a - \sin\frac{2N_1-1}{2} a}{2 \sin\frac{a}{2}} = \frac{\sin\frac{N_2-N_1}{2} a \ \cos\frac{N_1+N_2-1}{2} a}{\sin\frac{a}{2}}$$

If we set $a = 2\pi\varepsilon_n$ we have

$$\sum_{N_1}^{N_2-1} \sin 2\pi N\varepsilon_n = \frac{\sin (N_2 - N_1)\pi\varepsilon_n \sin (N_1 + N_2 - 1)\pi\varepsilon_n}{\sin \pi\varepsilon_n}$$

and

$$\sum_{N_1}^{N_2-1} \cos 2\pi N\varepsilon_n = \frac{\sin (N_2 - N_1)\pi\varepsilon_n \cos (N_1 + N_2 - 1)\pi\varepsilon_n}{\sin \pi\varepsilon_n}.$$

If we multiply the first of the last two equations by $\cos (g_n t + K_n)$ and the second by $\sin (g_n t + K_n)$ we get

$$\sum_{N_1}^{N_2-1} \sin \left(g_n t + K_n + \frac{g_n}{g} 2\pi N\right) =$$

$$= \frac{\sin (N_2 - N_1)\pi\varepsilon_n}{\sin \pi\varepsilon_n} \sin \left(g_n t + K_n + (N_1 + N_2 - 1)\pi\varepsilon_n\right)$$

or

$$\frac{1}{N_2 - N_1} \sum_{N_1}^{N_2-1} F(t + NP) = k \sin (gt + K) +$$

$$+ \sum_{1}^{n} k_n \frac{\sin (N_2 - N_1)\pi\varepsilon_n}{(N_2 - N_1) \sin \pi\varepsilon_n} \sin \left(g_n t + K_n + (N_1 + N_2 - 1)\pi\varepsilon_n\right). \quad (14)$$

The factor of reduction is thus

$$R = \frac{\sin (N_2 - N_1)\pi\varepsilon_n}{(N_2 - N_1) \sin \pi\varepsilon_n}, \quad (15)$$

which formula is the same as (11),

if we put

$$N_2 - N_1 = n \text{ and } \varepsilon_n = \frac{1}{P}. \quad (16)$$

If ε_n is zero or an integral number M, we have

$$\lim_{\varepsilon_n = M} \frac{\sin (N_2 - N_1)\pi\varepsilon_n}{(N_2 - N_1) \sin \pi\varepsilon_n} = \lim_{\varepsilon_n = M} \frac{\pi(N_2 - N_1) \cos (N_2 - N_1)\pi\varepsilon_n}{\pi(N_2 - N_1) \cos \pi\varepsilon_n} =$$

$$= \frac{(-1)^{M(N_2 - N_1)}}{(-1)^M} = (-1)^{M(N_2 - N_1 - 1)}$$

and

$$\sin (g_n t + K_n + (N_1 + N_2 - 1)\pi\varepsilon_n) = (-1)^{M(N_1 + N_2 - 1)} \sin (g_n t + K_n).$$

Thus

$$\frac{\sin (N_2 - N_1)\pi\varepsilon_n}{(N_2 - N_1)\sin \pi\varepsilon_n} \sin\left(g_n t + K_n + (N_1 + N_2 - 1)\pi\varepsilon_n\right) =$$

$$= (-1)^{2M(N_2-1)} \sin (g_n t + K_n) = \sin (g_n t + K_n).$$

But if $\varepsilon_n = M$ we have

$$\varepsilon_n = M = \frac{P - P_n}{P_n} \quad \text{or} \quad P_n = \frac{P}{M + 1},$$

that is, P_n is a fraction of P.

If $\varepsilon_n = 0$, P is equal to P_n.

We have thus proved the axiomatic proposition. that if we make an analysis corresponding to the period P, those terms which have periods equal to P or to a fraction of P are not changed.

If ε_n is neither zero nor an integral number but if $(N_2 - N_1)\varepsilon_n$ is an integer, say M,

we have

$$R = \frac{\sin (N_2 - N_1)\pi\varepsilon_n}{(N_2 - N_1)\sin \pi\varepsilon_n} = 0.$$

But in this case we have

$$M = (N_2 - N_1)\varepsilon_n = \frac{(N_2 - N_1)(P - P_n)}{P_n}$$

$$\text{or} \quad P_n(M + N_2 - N_1) = (N_2 - N_1)P.$$

As $(N_2 - N_1)P$ is equal to the whole observation period analysed we have proved the following theorem: if the analysed series of observations contains an integral number of periods which are not equal to P or to fractions of P the influence of the terms of these periods on the result of the analysis corresponding to the period P is zero.

In studying the maximal disturbing influence, it is only necessary to investigate the case when ε_n is nearly equal to zero or to an integer.

Put

$$\varepsilon_n = m + \varepsilon_n', \tag{17}$$

where m is equal to zero or to an integer and ε_n' is a small quantity.

We have then

$$\frac{k_n \sin (N_2 - N_1) \pi \varepsilon_n}{(N_2 - N_1) \sin \pi \varepsilon_n} \sin \left(g_n t + K_n + (N_1 + N_2 - 1) \pi \varepsilon_n\right) =$$

$$= \frac{k_n \sin (N_2 - N_1) \pi \varepsilon_n'}{(N_2 - N_1) \sin \pi \varepsilon_n'} \sin \left(g_n t + K_n + (N_1 + N_2 - 1) \pi \varepsilon_n'\right). \qquad (18)$$

For the factor of reduction (R) we then get

$$R = \frac{\sin (N_2 - N_1) \pi \varepsilon_n'}{(N_2 - N_1) \sin \pi \varepsilon_n'} = \frac{\sin (N_2 - N_1) \pi \varepsilon_n'}{(N_2 - N_1) \pi \varepsilon_n'}. \qquad (19)$$

This function has the following numerical maxima: (Page 13).

$(N_2 - N_1) \varepsilon_n'$	R
0	+ 1.000
1.43	— 0.218
2.46	+ 0.128
3.47	— 0.091
4.47	+ 0.071
5.48	— 0.058
6.48	+ 0.049
7.49	— 0.042
8.49	+ 0.037

The values of ε_n' are defined by the equation

$$\varepsilon_n = \frac{P - P_n}{P_n} = m + \varepsilon_n'. \qquad (20)$$

When we have a number of terms of appoximately equal periods, it is often convenient to express their sum by a sine-term of a period approximately equal to the mean of the respective periods. This sum we may call a "compound wave". Such compound waves can only be regarded as constant for a short time, but if the periods differ by small quantities only, the compound waves change but slowly.

Suppose that we desire to calculate a compound wave of the period P, and consisting of the terms $k_1 \sin(g_1 t + K_1)$, $k_2 \sin(g_2 t + K_2) \cdots$ $k_n \sin(g_n t + K_n)$, of the periods $P_1 = \frac{2\pi}{g_1}, \cdots P_n = \frac{2\pi}{g_n}$, where the different P_n are nearly equal to P. The compound wave is thus defined by the equation

$$k \sin(gt + K) =$$

$$= \sum_1^n k_n \frac{\sin(N_2 - N_1)\pi\varepsilon_n}{(N_2 - N_1)\pi\varepsilon_n} \sin\left[gt + K_n + (N_1 + N_2 - 1)\pi\varepsilon_n\right] \quad (21)$$

where N_1 is the number of periods $P = \frac{2\pi}{g}$ from the origin of time until the beginning of that time for which the compound wave is to be regarded as constant, N_2 is the corresponding number of periods P until the end of the same time and ε_n is defined by the equation

$$\varepsilon_n = \frac{P - P_n}{P_n}.$$

It is easy to see that this definition of a compound wave is in full harmony with the theory of mutual perturbations of the terms.

Correction of Wave-Lengths and Phase-Angles.

If in the series of observations a constant term $k \sin(gt + K)$ of the period $P = \frac{2\pi}{g}$ exists, and we want by trial to find out the period P, we must examine a number of periods P′, P″ … etc. in the vicinity of P and determine that value of P which gives the greatest amplitude. This determination, however, is somewhat uncertain, as a slight variation of P will not produce any appreciable effect on the magnitude of the amplitude. If we divide the material into several parts, each of which being large enough to give a determination of the term, we shall get a number of values of the constants k and K. It will then be necessary to have a formula for calculating the most probable values of P, k and K.

If we make an analysis of the curve $k \sin\left(\frac{2\pi}{P} t + K\right)$ corresponding to the period $P + \varDelta P$ instead of P, where $\varDelta P$ is a small quantity, we find different results for different parts of the analysed curve. Suppose that in the analysis of that part of the curve that corresponds to the time between the two moments $N_1(P + \varDelta P)$ and $N_2(P + \varDelta P)$, we have found the term

$$k' \sin\left(\frac{2\pi}{P + \varDelta P} t + K'\right),$$

N_1 and N_2 having the same values as in the last chapter.

If we put $$N_2 - N_1 = n,$$

we shall obtain, for all values of t in the vicinity of

$$T_1 = \frac{N_1 + N_2}{2}(P + \varDelta P),$$

$$k' \sin\left(\frac{2\pi}{P + \varDelta P} t + K'\right) = \frac{1}{n} \sum k \sin\left[\frac{2\pi}{P}\Big(t + m(P + \varDelta P)\Big) + K\right],$$

where the summation is to be taken from $m = -\frac{n-1}{2}$ to $+\frac{n-1}{2}$.

Thus

$$k' \sin\left(\frac{2\pi}{P + \varDelta P} t + K'\right) = k \sin\left(\frac{2\pi}{P} t + K\right) \frac{1}{n} \sum \cos\frac{2\pi m \varDelta P}{P} +$$

$$+ k \cos\left(\frac{2\pi}{P} t + K\right) \frac{1}{n} \sum \sin\frac{2\pi m \varDelta P}{P}.$$

But $$\sum_{-\frac{n-1}{2}}^{+\frac{n-1}{2}} \cos\frac{2\pi m \varDelta P}{P} = \frac{\sin\frac{\pi n \varDelta P}{P}}{n \sin\frac{\pi \varDelta P}{P}}$$

and $$\sum_{-\frac{n-1}{2}}^{+\frac{n-1}{2}} \sin\frac{2\pi m \varDelta P}{P} = 0.$$

If $\frac{\varDelta P}{P}$ is a small quantity, we have further

$$\frac{1}{P + \varDelta P} = \frac{1}{P} - \frac{\varDelta P}{P^2};$$

and
$$\sin \frac{\pi \varDelta P}{P} = \frac{\pi \varDelta P}{P}.$$

Thus

$$k' \sin \left(2\pi t \left(\frac{1}{P} - \frac{\varDelta P}{P^2}\right) + K'\right) = k R \sin \left(\frac{2\pi}{P} t + K\right), \quad (22)$$

where

$$R = \frac{\sin \frac{\pi n \varDelta P}{P}}{\frac{\pi n \varDelta P}{P}}. \quad (23)$$

The equation (22) must be satisfied by all values of t in the vicinity of T_1 and we must have

$$k' = k R \quad (24)$$

and
$$2\pi t \left(\frac{1}{P} - \frac{\varDelta P}{P^2}\right) + K' = \frac{2\pi}{P} t + K$$

or
$$K' - \frac{2\pi \varDelta P}{P^2} t = K$$

that is,
$$K - K' = \varDelta K' = - \frac{2\pi \varDelta P}{P^2} T_1 \quad (25)$$

As R is nearly equal to 1 and varies very slowly when $\frac{\varDelta P}{P}$ is a small quantity, the equation (24), or $k' =$ maximum, gives no good determination of $\varDelta P$ or P. The equation (25), however, gives a better determination of $\varDelta P$.

If we have divided our material into m parts and in the analysis of each of them have found the terms

$$k' \sin \left(\frac{2\pi}{P} t + K'\right), \quad k'' \sin \left(\frac{2\pi}{P} t + K''\right) \ldots k^{(m)} \sin \left(\frac{2\pi}{P} t + K^{(m)}\right),$$

we have the following equations of condition for the determination of $\varDelta P$:

$$\begin{aligned}\frac{2\pi\varDelta P}{P^2}(T'' - T') &= K'' - K' \\ \frac{2\pi\varDelta P}{P^2}(T''' - T') &= K''' - K' \\ \frac{2\pi\varDelta P}{P^2}(T^{(m)} - T') &= K^{(m)} - K',\end{aligned} \tag{26}$$

where the $T^{(m)}$ are equal to the middle of the time of the analysed curve.

Having made an analysis corresponding to the period P, we find the corrected value P_1 of the period from the equation

$$P_1 = P - \varDelta P. \tag{27}$$

The corrected value K of the phase-angle we find from the equations

$$\begin{aligned}K &= K' - \frac{2\pi\varDelta P}{P^2}T' \\ K &= K'' - \frac{2\pi\varDelta P}{P^2}T'' \\ K &= K^{(m)} - \frac{2\pi\varDelta P}{P^2}T^{(m)}\end{aligned} \tag{28}$$

or

$$K = \frac{K' + K'' + \cdots + K^{(m)}}{m} - \frac{2\pi\varDelta P}{P^2}\,\frac{T' + T'' + \cdots + T^{(m)}}{m}. \tag{28a}$$

Application to the Temperature-Curve of Uppsala 1880—1912.

In order to make an application of this theory to a practical problem, I have used the temperature-observations of Uppsala 1880—1912. The theory gives us a method to decide whether constant temperature-waves of the character of sine-terms exist in the temperature-variations, except the daily and yearly variations, i. e. whether a part of the irregular variations can be developed in a trigonometrical series of constant terms. In Uppsala observations of the air-temperature and other meteorological phenomena have been registrered for every hour ever since the year 1866[1]). The daily variation is eliminated by taking means of the observations for each hour of each day, and the yearly one by subtracting a calculated yearly variation computed from a series of observations comprising a great number of years. In this analysis I have not subtracted the yearly variation, but in every case I have analysed a series of observations containing an integral number of the examined period. Moreever each such series begins and ends as near the 1st of January as possible. It is easy to see that the influence of the yearly variation on the determined terms can be neglected (Page 17). I have hitherto examined all the periods (wave-lengths) between 20 and 40 days, with an interval in wave-length of from 0.02 to 0.05 days in order to find those periods that give the greatest variation (amplitude).

The analysis corresponding to the period P was made in the following way. The temperature-curve (mean temperature for each day) was smoothed out by forming three days mean for each day in order to eliminate the variations of short duration. *The origin of time is in this investigation throughout 1900 Jan. 0.* (Mean European time). 1900 Jan. 1st has the number + 1, 1912 Jan. 1st = + 4383, 1880 Jan. 1st = − 7304, etc.

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[1]) Bulletin mensuel de l'observatoire météorologique de l'université d'Uppsala.

Starting from this origin of time multiples of the period P are set off on the curve of observations, then the ordinates of the curve are read at 8, 10 or 12 equidistant points within each interval P. The average values are then calculated separately for each part of such an interval. In this way 8, 10 or 12 average values are obtained, and we have to calculate a sine-term which corresponds as nearly as possible to the variation of these averages. With periods between 19.95 and 21.74 days the period is divided into 8 parts, with periods between 21.76 and 26.24 days into 10 parts, and with periods 26.26 to 40.85 days into 12 parts.

In order to reduce the great labour of making a separate calculation of each period, an original analysis was only made for periods equal to 20.0, 21.0, 22.5, 24.0, 25.5 ... 39.0 and 40.5 days, i. e. with an interval of 1.5 days. From the readings corresponding to the period P we can obtain the readings corresponding to another period in the vicinity of P by a renewed addition in connection with a small displacement in the tables. I have thus calculated the variations corresponding to each period with an interval of 0.1 days for periods shorter than 30 days and with an interval of 0.25 days for periods longer than 30 days. Finally in the results for the different years I have made a further addition with displacements and thus obtained the variations corresponding to all periods with an interval of 0.02 days for periods shorter than 30 days and with an interval of 0.05 days for longer periods.

It is to be observed that two consecutive sine-curves with these small differences in wave-length are not independent of each other. The series of observations 1880—1912 contains almost exactly 12000 days and there are thus 600 periods of 20 days, 400 periods of 30 days and 300 periods of 40 days. For periods nearly equal to 20 days, the periods must differ by 0.0333 days in order to separate exactly once, and the two terms are then independent of each other (Page 17). The corresponding difference in wave-length is 0.075 days for periods equal to 30 days, and 0.133 days for periods equal to 40 days.

In this way we can find by trial those periods that give the greatest variation (amplitude); the probability of the existence of such a sine-term is thus greater than the probability of the existence of a term of somewhat greater or smaller period (Equ. 9).

In order to make sure of the real existence of the computed sine-terms in the curve analysed, we can 1) examine the magnitude of the amplitudes in comparison with their mean errors; 2) examine whether the term found is of a constant nature for different parts of the curve analysed; 3) investigate whether a computed term occurs before 1880 and after 1912.

To find the mean error of an observation the most practical method is first to subtract the yearly variation and then to calculate a smooth curve representing the differences between the actual mean temperature of the day and the normal temperature, the shortest variations being eliminated. The normal temperature for Uppsala used in this investigation has been calculated from the observations for 1866—1912 and is given in Table 3. If for instance we use 10 days' mean for each day the variations within a period of 10 days or less are almost eliminated, and the amplitude of a sine-term of a period of 20 days is reduced by the factor $R_{1/2} = 0.637$ (see Table 1) For periods equal to 30 and 40 days the factors of reduction are respectively $R_{1/3} = 0.83$ and $R_{1/4} = 0.89$. For the years 1910, 1911 and 1912 the mean error corresponding to 10 days' means in the differences named is nearly constant and has the values $\varepsilon = \pm 1^0.68$, $\pm 1^0.70$ and $\pm 1^0.60$ C. If we assume this value to be $\pm 1^0.70$, the mean errors in the amplitudes of the sine-terms resulting from the analysis corresponding to the period P can be calculated from the formula

$$\varepsilon_k^2 = \frac{2\varepsilon^2 - k^2}{R^2 N}$$

$$R_{\frac{n}{P}} = \frac{\sin\frac{\pi n}{P}}{\frac{\pi n}{P}}\,.$$

ε_k in Equ. (9) must be divided by R as the analysed curve is formed by using 3 days' means instead of 10 days' means and the former operation has no appreciable effect on k. The forming of 3 days' means is made only in order to make it possible to reduce the number of equations of condition. ε_k can then be compared with the amplitude k. We have thus, if k is a small quantity:

Period:	P	$=$ 20 days	30 days	40 days
Number of observations:	N	$=$ 8 · 600	12 · 400	12 · 300
Mean error of an observation:	ε	$= \pm 1^0.70$	$\pm 1^0.70$	$\pm 1^0.70$
Factor of reduction:	$R_{\frac{n}{P}}$	$=$ 0.64	0.83	0.89
Mean error of the amplitudes:	ε_k	$= \pm 0^0.054$	$\pm 0^0.042$	$\pm 0^0.045$

The mean error of an observation is if we use 5 days' means for each day:

		1910	1911	1912
ε	=	$\pm 3^0.01$	$\pm 3^0.34$	± 3.24

The factors of reduction in this case (Table 1) are

$$R_{1/4} = 0.89, \qquad R_{1/6} = 0.95, \qquad R_{1/8} = 0.97.$$

We have thus:

P	=	20 days	30 days	40 days
N	=	4800	4800	3600
ε	=	$\pm 3^0.3$	$\pm 3^0.3$	$\pm 3^0.3$
$R_{\frac{n}{p}}$	=	0.89	0.95	0.97
ε_k	=	$\pm 0^0.076$	$\pm 0^0.071$	$\pm 0^0.080$

If we use the mean temperature of each day we shall find that the mean error in the difference between the mean temperature and the normal temperature is somewhat less than $\pm 5^0.0$. But if we had used the mean temperature of each day withouth calculating means of some days' temperature, it would have been necessary to use each one of these observations when calculating the terms. The number of observations would then have been 12000, the factors of reduction would have been 1.000 and the mean error of the amplitude would then be:

$$\varepsilon_k = \frac{5^0.0}{\sqrt{6000}} = \pm 0^0.065.$$

As an example of the methods mentioned of proving the existence of a term, I shall use the terms with the periods 19.96 and 20.06 days. From the observations 1880—1912 in total I found maximal amplitudes corresponding to these periods, viz. the terms $0^0.29 \sin\left(\frac{2\pi}{19.96} t + 29\right)$ and $0^0.23 \sin\left(\frac{2\pi}{20.06} t + 245\right)$. I then divided the material into three parts in order to investigate whether I could trace the terms in each of these parts, and later on I made an analysis of the observations for the period 1870—1879. The wave-length is divided into 8 equal parts and for each one of these parts I have calculated the average value of all observations during a period of 10 (13) years and found that the variations represented by sine-terms are almost constant in nature, and that the phase-angles increase in proportion to the time. (Fig. 1 and 2.)

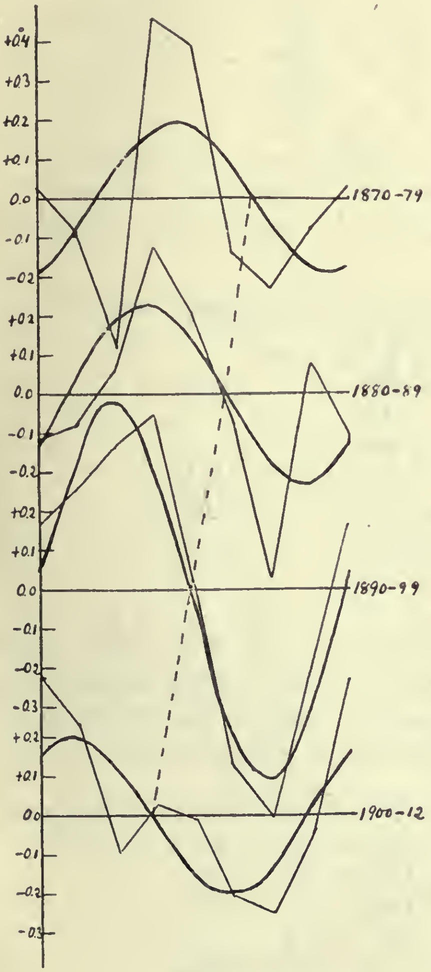

Fig. 1.

Variation of Uppsala temperature in the period 19.96 days. The linear change of the phase-angle corresponds to an error in the wave-length of — 0.011 days.

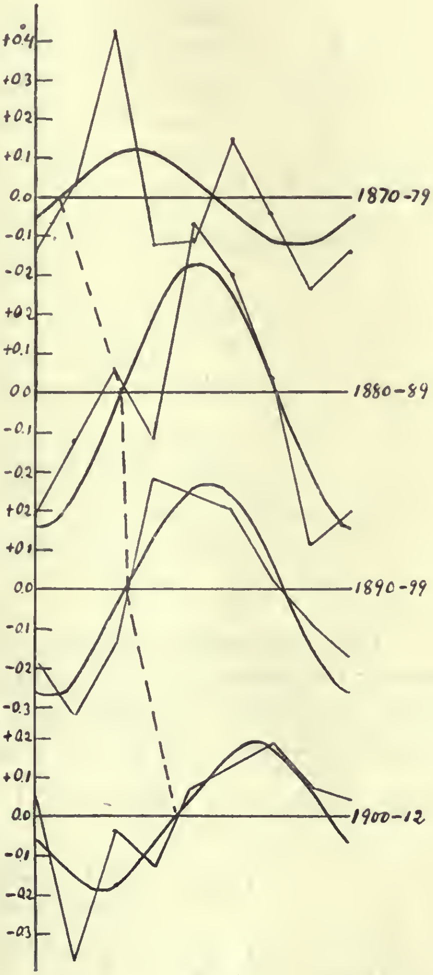

Fig 2.

Variation of Uppsala temperature in the period 20.06 days. The nearly linear change of the phase-angle corresponds to an error in the wave-length of + 0.014 days.

$P = 19.96$ days. Fig. 1.

Observed Variation of the Average Values.

	1870—79	1880—89	1890—99	1900—1912
$^0/_8$	$+0^0.03$	$-0^0.11$	$+0^0.17$	$+0^0.35$
$^1/_8$	-0.08	-0.08	$+0.26$	$+0.22$
$^2/_8$	-0.37	$+0.07$	$+0.37$	-0.10
$^3/_8$	$+0.46$	$+0.38$	$+0.45$	$+0.02$
$^4/_8$	$+0.39$	$+0.20$	$+0.01$	$+0.01$
$^5/_8$	-0.14	-0.07	-0.45	-0.21
$^6/_8$	-0.23	-0.47	-0.58	-0.25
$^7/_8$	-0.08	$+0.08$	-0.20	-0.05

From these residuals we derive:

$$1870\text{—}79: \quad 0^0.19 \sin\left(\frac{2\pi}{19.96}\, t + 292^0\right)$$

$$1880\text{—}89: \quad 0.23 \sin\left(\frac{2\pi}{19.96}\, t + 324\right)$$

$$1890\text{—}99: \quad 0.48 \sin\left(\frac{2\pi}{19.96}\, t + 6\right)$$

$$1900\text{—}12: \quad 0.20 \sin\left(\frac{2\pi}{19.96}\, t + 50\right)$$

The phase-angles 292, 324, 366 and 410 form an increasing series nearly proportional to the differences of the corresponding times 1875, 1885, 1895 and 1906.5 (See Fig. 1). These values of the phase-angles give us a value for a correction of the wave-length. The formula (26) gives $\left(\frac{2\pi}{P^2} = 0^0.904\right)$ the following equations of condition:

$$\begin{aligned} 0.904 \cdot 3652\, \Delta P &= 32 \\ 0.904 \cdot 7305\, \Delta P &= 74 \\ 0.904 \cdot 11505\, \Delta P &= 118 \end{aligned}$$

Hence

$$\Delta P = +0.011 \text{ days,}$$

and the corrected value of P (Equ. 27)

$$P = 19.96 - 0.011 = 19.949 \text{ days.}$$

Equation (28) gives a value for the correction ΔK of the phase-angle K. We must then observe that T′ etc. must be reckoned from 1900.0, i. e.

$$\begin{aligned} T' &= -9131 \text{ days} \\ T'' &= -5478 \\ T''' &= -1826 \\ T^{IV} &= +2374 \end{aligned}$$

We have thus

$$\begin{aligned} K' &= 292 + 92 = 24^0 \\ K'' &= 324 + 55 = 19 \\ K''' &= 6 + 18 = 24 \\ K^{IV} &= 50 - 24 = 26 \end{aligned}$$

$$\text{Mean: } K = 23^0.$$

For the value of k we may take the mean of the four amplitudes 0.19, 0.23, 0.48 and 0.20,

$$\text{or} \quad k = 0^0.27.$$

We have then found the term

$$0^0.27 \sin\left(\frac{2\pi}{19.946}\, t + 23^0\right).$$

P = 20.06 days. Fig. 2.

Observed Variation of the Average Values.

	1870—79	1880—89	1890—99	1900—1912
$^0/_8$	$-0^0.14$	$-0^0.31$	$-0^0.17$	$+0^0.05$
$^1/_8$	$+0.03$	-0.12	-0.32	-0.36
$^2/_8$	$+0.42$	$+0.06$	-0.14	-0.04
$^3/_8$	-0.12	-0.12	$+0.28$	-0.13
$^4/_8$	-0.11	$+0.44$	$+0.24$	$+0.07$
$^5/_8$	$+0.15$	$+0.40$	$+0.20$	$+0.13$
$^6/_8$	-0.04	$+0.04$	$+0.03$	$+0.19$
$^7/_8$	-0.23	-0.39	-0.09	$+0.08$

We get from these residuals:

$$1870\text{—}79: \quad 0^0.12 \sin\left(\frac{2\pi}{20.06}\, t + 336^0\right)$$

$$1880\text{—}89: \quad 0.33 \sin\left(\frac{2\pi}{20.06}\, t + 263\right)$$

$$1890\text{—}99: \quad 0.27 \sin\left(\frac{2\pi}{20.06}\, t + 255\right)$$

$$1900\text{—}12: \quad 0.19 \sin\left(\frac{2\pi}{20.06}\, t + 198\right)$$

Hence (Equ. 26)

$$\Delta P = -0.014.$$

Thus (Equ. 27)

$$P = 20.06 + 0.014 = 20.074.$$

Further

$$\begin{aligned} K' &= 336 - 114 = 222^0 \\ K'' &= 263 - 68 = 195 \\ K''' &= 255 - 23 = 232 \\ K^{IV} &= 198 + 30 = 228 \end{aligned}$$

$$\text{Mean:}\quad K = 220.$$

The most probable value of the constants are then

$$0^0.23 \sin\left(\frac{2\pi}{20.074} t + 220^0\right).$$

At the analysis of the temperature-curve 1880—1912, I found the same maximal amplitude $0^0.23$ for the two periods 20.06 and 20.08 days, viz. the terms

$$0^0.23 \sin\left(\frac{2\pi}{20.06} t + 245\right) \quad \text{and} \quad 0.23 \sin\left(\frac{2\pi}{20.08} t + 213\right).$$

The true walue of the wave-length (20.074) lies thus between these two periods, and we see further that the condition, amplitude equal to maximum, only gives an appoximated value of the wavelength.

The period of observation 1870—1912 contains 784 periods equal to 20 days. The number of observations is thus $8 \cdot 784 = 6272$. The mean error of the amplitudes is if we start from 5 days mean temperature ($\varepsilon = \pm 3^0.3$).

$$\varepsilon_k = \frac{3^0.3}{\sqrt{3136 \cdot 0.89}} = \pm 0^0.067.$$

It is thus probable that the two terms with the periods 19.949 and 20.074 days really exist in the temperature, at least as a mean during the period of a decennium, but we do not know whether these terms are always there. The amplitudes can for instance change from time to time, but such variations, when they are of a regular character, can be expressed by a sum of new terms with nearly the same periods.

As the temperature-variations are of a much complicated nature we can expect that a very large number of terms must exist. It is thus necessary to determine all wave-lengths for which the amplitudes are a maximum. I have examined all wave-lengths between 19.94 and 40.85 days, and then found that no term of a period between these numbers exists with greater amplitude than 0°.38 C. (total effect 0°.76), and that there possibly is a great number of terms with amplitudes between 0°.15 and 0°.37. 1) The terms are well separated by intervals corresponding to vanishing amplitudes (round 0°.05), which correspond to the mean error. I have not generally calculated all intermediate amplitudes, but a glance at the small variations of the different average values has proved, that the resulting amplitudes are very small. In fig. 3 we have a survey of all computed variations corresponding to periods between 29 and 30 days, which will give a good view of the method and of the reality of the terms. In order to facilitate the calculations of the amplitudes and phase-angles I have subtracted a variation from the found residuals corresponding to periods equal to the half and the third part of the original period. We can then draw that sine-line which best represents the variation. In this interval wee see that there are four prominent maxima of amplitudes, viz. the periods 29.38, 29.60, 29.78 and 29.94 days, corresponding to amplitudes greater than 0°.20. I have then divided the material into two parts and proved that the results of the analysis is nearly the same for these two parts. From the differences of the phase-angles I have determined a small correction to the wave-lengths and found that the most exact values of the wave-lengths are: 29.380, 29.602, 29.792 and 29.947 days (See the List of the terms!)

For all maximal amplitudes greater than 0°.14. I have in this way divided the material into two parts (1880—1894 and 1895—1912) and computed the corrections of the wave-lengths and of the phase-angles, according to Equ. (26). We shall see the results in the following table.

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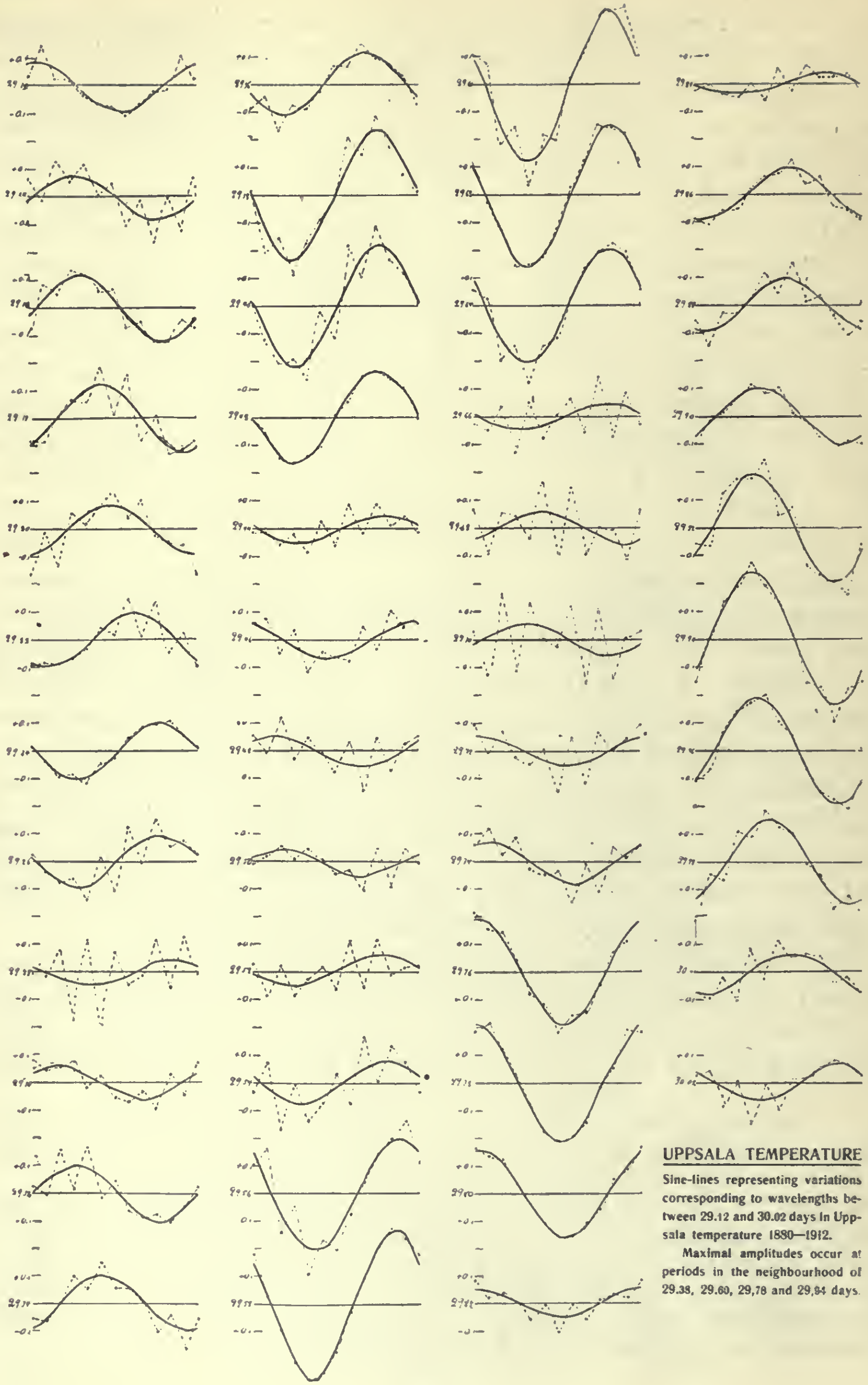
1) In my previous treatice I had found a few term with greater amplitudes, the periods of which was $P_{\delta+2a}$, $P_{\delta+\frac{a}{2}}$ and $P_{\delta-\frac{a}{2}}$. These terms was computed from the periods $P_{\delta+a}$ and P_{δ} by renewed addition in connection with diplacements in the tabulated observations. At the diplacements certain observations were omitted, but in these cases the omitted observations corresponded exactly to the yearly period. The yearly variation was then not fully eliminated and the resulting terms became erroneous. At the later investigation no such disturbing influence will take place.



UPPSALA TEMPERATURE

Sine-lines representing variations corresponding to wavelengths between 29.12 and 30.02 days in Uppsala temperature 1880—1912.

Maximal amplitudes occur at periods in the neighbourhood of 29.38, 29.60, 29,78 and 29,94 days.

Fig. 3.

Table 2.

List of terms of periods between 19.94 and 40.85 days
in Uppsala temperature 1880—1912.

Examined wave-length		Amplitudes and phase-angles for different epochs			Corrected amplitudes and phase-angles	Correction of wave-length	Corrected wave-length
	1)	**80-89;**	**90-99;**	**00-12;**	2)		
19.96 days	A	0°.23	0°.48	0°.20	0°.29		
	Ph	324°	6°	50°	20°	— 0.010 days	19.950 days 2)
		80-94	**95-12**		3)		
20.08	A	0.24	0.22		0.23		
	Ph	168	228		213	— 0.011	20.069 3)
20.12	A	0.24	0.20		0.22		
	Ph	336	258		277	+ 0.014	20.134
20.18	A	0.17	0.13		0.15		
	Ph	72	132		117	— 0.011	20.169
20.26	A	0.09	0.23		0.16		
	Ph	282	228		241	+ 0.010	20.270
20.34	A	0.08	0.20		0.14		
	Ph	90	84		85	+ 0.001	20.341
20.48	A	0.17	0.18		0.18		
	Ph	192	216		210	— 0.004	20.476
20.56	A	0.22	0.15		0.18		
	Ph	132	150		146	— 0.003	20.557
20.70	A	0.23	0.14		0.18		
	Ph	72	36		45	+ 0.007	20.707
20.78	A	0.24	0.10		0.17		
	Ph	348	270		289	+ 0.014	20.794
20.86	A	0.21	0.21		0.21		
	Ph	150	96		109	+ 0.010	20.870
20.94	A	0.13	0.20		0.17		
	Ph	66	354		12	+ 0.014	20.954

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1) A = amplitude.
Ph = phase-angle.

2) See pag. 29.

3) See pag. 30.

Examined wave-length		Amplitudes and phase-angles for different epochs		Corrected amplitudes and phase-angles	Correction of wave-length	Corrected wave-length
		80-94	95-12			
21.00 days	A	0°.20	0°.23	0°.22		
	Ph	243°	216°	223°	+ 0.005 days	21.005 days
21.10	A	0.08	0.18	0.13		
	Ph	282	240	250	+ 0.008	21.108
21.20	A	0.10	0.17	0.14		
	Ph	210	240	233	— 0.006	21 194
21.26	A	0.34	0.05			
	Ph	84	336			
21.40	A	0.20	0.26	0 23		
	Ph	162	228	212	— 0.013	21 387
21.52	A	0.13	0.18	0.16		
	Ph	222	216	217	+ 0.001	21.521
21.58	A	0.05	0.39	0.22		
	Ph	294	6	349	— 0.015	21.565
21.68	A	0.36	0.04	0.20		
	Ph	120	120	120	0.000	21.680
21.72	A	0.33	0.12	0.22		
	Ph	336	312	318	+ 0.005	21.725
21.82	A	0.15	0.34	0.24		
	Ph	48	126	107	— 0.017	21.803
21.88	A	0.30	0.16	0.23		
	Ph	270	300	293	— 0.007	21.873
21.94	A	0.27	0.20	0.23		
	Ph	84	42	52	+ 0.009	21.949
22 02	A	0.14	0.15	0.15		
	Ph	348	312	321	+ 0.008	22.028
22.14	A	0.30	0.24	0.27		
	Ph	150	168	164	— 0.004	22.136
22.28	A	0.11	0.15	0.13		
	Ph	156	210	197	— 0.012	22.268
22.34	A	0.21	0.25	0.23		
	Ph	210	210	210	0.000	22.340
22.44	A	0.30	0.25	0.27		
	Ph	18	48	41	— 0.007	22.433

Examined wave-length		Amplitudes and phase-angles for different epochs		Corrected amplitudes and phase-angles	Correction of wave-length	Corrected wave-length
		80-94	95-12			
22.48 days	A	0°.18	0°.18	0°.18		
	Ph	240°	174°	190°	+ 0.015 days	22.495 days
22.56	A	0.13	0.18	0.16		
	Ph	72	138	122	— 0.016	22.544
22.62	A	0.19	0.08	0.13		
	Ph	216	240	234	— 0.006	22.614
22.84	A	0.18	0.20	0.19		
	Ph	288	324	315	— 0.008	22.832
23.02	A	0.19	0.20	0.20		
	Ph	24	24	24	0.000	23.020
23.10	A	0.24	0.38	0.31		
	Ph	0	318	328	+ 0.010	23.110
23.16	A	0.06	0.30	0.18		
	Ph	60	48	51	+ 0.003	23.163
23.28	A	0.22	0.16	0.19		
	Ph	288	336	324	— 0.011	23.269
23.38	A	0.36	0.05	0.20		
	Ph	96	60	69	+ 0.008	23.388
23.44	A	0.22	0.42	0.32		
	Ph	258	198	213	+ 0.014	23.454
23.50	A	0.15	0.31	0.23		
	Ph	12	330	340	+ 0.010	23.510
23.62	A	0.20	0.18	0.19		
	Ph	336	312	318	+ 0.006	23 626
23.72	A	0.10	0.24	0.17		
	Ph	246	210	219	+ 0.010	23.730
23.86	A	0.26	0.10	0.18		
	Ph	264	288	282	— 0.006	23.854
23.94	A	0 33	0.04	0.18		
	Ph	18	54	27	—.0.010	23.930
24.14	A	0.32	0.06	0.19		
	Ph	174	180	179	– 0.002	24.138
24.36	A	0.26	0.08	0.17		
	Ph	306	6	351	-- 0.016	24.344

Examined wave-length		Amplitudes and phase-angles for different epochs		Corrected amplitudes and phase-angles	Correction of wave-length	Corrected wave-length
		80-94	95-12			
24.44 days	A	0°.16	0°.26	0°.21		
	Ph	354°	60°	44°	— 0.018 days	24.422 days
24.50	A	0.40	0.19	0.29		
	Ph	120	96	102	+ 0.006	24.506
24.70	A	0.30	0.19	0.24		
	Ph	126	96	103	+ 0.008	24.708
24.82	A	0.17	0.20	0.18		
	Ph	330	246	266	+ 0.024	24.844
24.98	A	0.13	0.17	0.15		
	Ph	48	42	43	+ 0.002	24.982
		66-89	90-12			
25.12	A	0.15	0.19	0.14		
	Ph	72	32	37	+ 0.008	25.128
		80-94	95-12			
25.28	A	0.22	0.11	0.15		
	Ph	228	276	264	— 0.014	25.266
25.40	A	0.22	0.14	0.18		
	Ph	138	120	124	+ 0.005	25.405
26.60	A	0.26	0.10	0.18		
	Ph	96	126	119	— 0.009	25.591
25.78	A	0.14	0.34	0.24		
	Ph	156	144	147	+ 0.004	25.784
25.86	A	0.32	0.32	0.32		
	Ph	162	198	189	— 0.011	25.849
26.00	A	0.13	0.27	0.20		
	Ph	174	132	142	+ 0.013	26.013
26.32	A	0.15	0.31	0.23		
	Ph	144	150	149	— 0.002	26.318
26.46	A	0.09	0.22	0.16		
	Ph	210	150	165	+ 0.019	26.479
26.60	A	0.10	0.23	0.17		
	Ph	324	0	351	— 0.012	26.588
26.66	A	0.23	0.20	0.21		
	Ph	66	84	80	— 0.006	26.654

Examined wave-length		Amplitudes and phase-angles for different epochs		Corrected amplitudes and phase-angles	Correction of wave-length	Corrected wave-length
		80-94	**95-12**			
26.76 days	A	0°.26	0°.16	0°.21		
	Ph	198°	198°	198°	0.000 days	26.760 days
26.90	A	0.06	0.19	0.13		
	Ph	138	78	93	+ 0.020	26.920
27.00	A	0.12	0.23	0.18		
	Ph	300	24	4	— 0.028	26.972
27.10	A	0.24	0.28	0.26		
	Ph	306	294	297	+ 0.004	27.104
27.18	A	0.19	0.20	0.20		
	Ph	234	252	248	— 0.006	27.174
27.30	A	0.19	0.19	0.19		
	Ph	204	198	199	+ 0.002	27.302
27.50	A	0.10	0.23	0.17		
	Ph	174	192	188	— 0.006	27.494
27.70	A	0.27	0.23	0.25		
	Ph	228	210	214	+ 0.007	27.707
28.02	A	0.14	0.14	0.14		
	Ph	174	174	174	0.000	28.020
28.12	A	0.42	0.15	0.28		
	Ph	336	300	309	+ 0.013	28.133
28.40	A	0.15	0.16	0.16		
	Ph	144	120	126	+ 0.009	28.409
28.50	A	0.25	0.10	0.17		
	Ph	216	210	211	+ 0.002	28.502
28.58	A	0.20	0.12	0.16		
	Ph	30	36	35	— 0.002	28.578
28.84	A	0.06	0.22	0.14		
	Ph	18	60	50	— 0.016	28.824
28.92	A	0.15	0.25	0.20		
	Ph	114	78	87	+ 0.014	28.934
29.38	A	0.19	0.28	0.24		
	Ph	183	183	183	0.000	29.380

Examined wave-length		Amplitudes and phase-angles for different epochs		Corrected amplitudes and phase-angles	Correction of wave-length	Corrected wave-length
		66-89	**90-12**			
29.60 days	A	0°.24	0°.35	0°.29		
	Ph	155°	152°	150°	+ 0.002 days	29.602 days
		80-94	**95-12**			
29.78	A	0.25	0.18	0.21		
	Ph	96	66	73	+ 0.012	29.792
29.95	A	0.26	0.20	0.23		
	Ph	311	320	318	— 0.003	29.947
30.20	A	0 27	0.20	0.23		
	Ph	108	60	72	+ 0.020	30.220
30.65	A	0.17	0 14	0.15		
	Ph	234	228	229	+ 0.002	30.652
30.80	A	0.20	0.31	0.26		
	Ph	12	348	354	+ 0.010	30.810
30.95	A	0.26	0.28	0.27		
	Ph	0	0	0	0.000	30.950
31.15	A	0.27	0.22	0.24		
	Ph	306	276	283	+ 0.013	31.163
31.35	A	0.30	0.21	0.25		
	Ph	168	162	164	+ 0.002	31.352
31.85	A	0.15	0.30	0.23		
	Ph	156	120	129	+ 0.017	31.867
32.15	A	0.17	0.21	0.19		
	Ph	198	222	216	— 0.011	32.139
32.30	A	0.14	0.30	0.22		
	Ph	78	78	78	0.000	32.300
32.50	A	0.27	0.20	0.23		
	Ph	324	312	315	+ 0.006	32.506
32.85	A	0.28	0.12	0.20		
	Ph	150	210	195	— 0.028	32.822
33.20	A	0.21	0.19	0.20		
	Ph	264	240	246	+ 0.012	33.212
33.35	A	0.27	0.12	0.19		
	Ph	78	78	78	0.000	33.350
33.65	A	0.26	0.31	0.29		
	Ph	102	162	147	— 0.032	33.618

Examined wave-length		Amplitudes and phase-angles for different epochs		Corrected amplitudes and phase-angles	Correction of wave-length	Corrected wave-length
		80-94	95-12			
33.90 days	A	0°.28	0°.10	0°.19		
	Ph	228°	204°	219°	+ 0.013 days	33.913 days
34.05	A	0.23	0.21	0.22		
	Ph	330	294	303	+ 0.019	34.069
35.05	A	0.20	0.19	0.19		
	Ph	48	60	57	— 0.007	35.043
35.25	A	0.26	0.30	0.28		
	Ph	84	126	116	— 0.023	35.227
35 65	A	0.33	0.12	0.22		
	Ph	258	216	226	+ 0.025	35.675
36.15	A	0.43	0.20	0.31		
	Ph	78	24	37	+ 0.032	36.182
36.50	A	0.32	0.31	0.31 [1]		
	Ph	204	264	249	— 0.036	36.464 [1]
37.25	A	0.14	0.20	0.17		
	Ph	192	258	242	— 0.042	37.208
37.90	A	0.24	0.11	0.17		
	Ph	168	180	177	— 0.008	37.892
38.30	A	0.17	0.21	0.19		
	Ph	222	204	208	+ 0.012	38.312
38.50	A	0.26	0.23	0.24		
	Ph	324	0	351	— 0.025	38.475
38.70	A	0.13	0.26	0.20		
	Ph	30	60	53	— 0.021	38.679
38.90	A	0.25	0.39	0.32		
	Ph	96	72	78	+ 0.017	38.917
39.40	A	0.40	0.29	0.34		
	Ph	120	96	102	+ 0.017	39.417
40.15	A	0.39	0.33	0.36 [2]		
	Ph	60	96	87	— 0.027	40.123 [2]
40.35	A	0.14	0.45	0.30		
	Ph	174	180	179	— 0.004	40.346
40.50	A	0.23	0.32	0.28		
	Ph	276	240	249	+ 0.027	40.527

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[1] See pag. 46. [2] See pag. 49.

We se from the table that it is only in one case that the results from the two parts of the material of observations 1884—94 and 1895—1912 are contradictory, i. e. when the phase-angles differ by more than 90°. This occurs at the term of the period 21.26 days, the phase-angles showing a difference of 98°. In the vicinity of this wave-length there is probably no real term. Some of the other terms of small amplitudes are uncertain and cannot be proved to be real. We can, however, draw the following conclusions: 1) If constant sine-terms exist of periods between 20 and 40 days in the temperature-variations of Uppsala during the time 1880—1912, there is possibly a great number of them to be found with amplitudes between 0°.14 and 0°.37; 2) if a sine-term is determined from the observations 1880—1912 in that way, that its decreasing effect on the mean error is a maximum and its amplitude is greater than 0°.14, such a term decreases the mean error in both of the two halves of the material with a probability of about $\frac{111}{112}$.

A study of interest is to find out eventual periods of repetition, i. e. "periods" in the usual sence of the word. It we add two terms of the wave-lengths P_1 and P_2 their sum represents a wave-line of a period P determined by the equation

$$k_1 \sin\left[\frac{2\pi}{P_1} t + K_1\right] + k_2 \sin\left[\frac{2\pi}{P_2} t + K_2\right] =$$

$$= k_1 \sin\left[\frac{2\pi}{P_1}(t+P) + K_1\right] + k_2 \sin\left[\frac{2\pi}{P_2}(t+P) + K_2\right].$$

Hence

$$\cos\frac{2\pi P}{P_1} = \cos\frac{2\pi P}{P_2} = 1, \quad \text{and} \ \sin\frac{2\pi P}{P_1} = \sin\frac{2\pi P}{P_2} = 0.$$

Thus

$$\frac{2\pi P}{P_1} = 2m\pi \quad \text{and} \quad \frac{2\pi P}{P_2} = 2n\pi,$$

$$\text{or} \quad P = mP_1 = nP_2,$$

where m and n are integers.

The m and n can be found by expressing $\frac{P_1}{P_2}$ in a continued fraction.

For two nearly equal periods P_1 and P_2 we can find the smallest period of repetition by putting $m = n \pm 1$.

We then get $$\frac{1}{P} = \left|\frac{1}{P_1} - \frac{1}{P_2}\right|.$$

The sum of the terms of the periods 29.380 and 29.602 days has thus the period 3920 days = 10.74 years, the sum of the term of the periods 29.602 and 29.792 days has the period 12.71 years etc. The sum of the terms of the periods 19.949 and 20.074 days corresponds to the period 8.77 years. A great number of consecutive terms corresponds to periods of repetition nearly equal to 11 years. A study of the inverse values of the periods shows that the periods of repetition 19, 14, 11 and 9 years are principally represented.

This, however, is a very uncertain method of finding eventual periods of repetition, as the determination of the wave-lengths is not sufficiently good for this purpose. On the other hand it is probable that the constants of the different terms: amplitudes, wave-lengths and phase-angles, are not independent of each other, and that it is possible to reduce the number of independent constants by a number of relations between the said quantities. Such relations would give us a method of getting a better determination of the constants and make it possible to use the terms for a wider extrapolation. As at the extrapolation a little error in the wave-length gives rise to a large error in the phase of the sine-term, it is difficult to use the empirical wave-lengths at the predicting of the temperature. This error cannot be reduced by taking observations for a longer time. The constants would then be better determined, but on the contrary we must extrapolate from a more distant time, as the found constants are valid for the mean of the time of observations. It is necessary to use the most accurate technic in determining the terms, especially their wave-lengths, if an extrapolation is intended. For instance an error of only 0.02 days at a period of 20 days makes at the extrapolation an error of

$$\frac{0.02\,(1913.0 - 1880.0) \cdot 365.24}{2 \cdot 20} = 6 \text{ days},$$

which error of the phase gives a contradictory wave, as 6 is more than the fourth part of 20. The extrapolation can thus not give us a general proof of the reality of the terms.

Compound Waves.

We have seen that the terms, which we have calculated from the observations, manifest themselves as a greater variation of the means of the temperature as it probably would be if the variation of the temperature were quite irregular. As the terms in the table corresponds to nearly equal wave-lengths it is necessary to use observations from 10 or 15 years to get the terms isolated. If we use observations from a shorter time we do not get the pure terms, but a sum of the effects of a number of adjacent periods.

If we want to prove the existence of a term during a shorter time than 10 years we must calculate the "compound wave" defined by the equation (21) and compare this one with the observed variation, calculated directly from the observations.

For this purpose I have calculated the compound waves with the wavelengths 29.600, 36.464 and 40.123 days for different 5 years periods and the compound wave with the wave-length 28.50 days (arbitrary chosen period) for each year 1890—1909 and compared them with the observed variation corresponding to the same periods.

Compound wave.

Period: 29.600 days. Fig. 4.

	P = 29.380		29.602		29.792		29.947		Sum.
	K+ΔK,	R	K+ΔK,	R	K+ΔK,	R	K+ΔK	R	P = 29.600
1866-69:	197°,	0.79	171°,	1.00	263°,	0.85	168°,	0.55	$0^\circ.25 \sin(gt+191^\circ)$
1870-74:	348,	0.68	169	„	134,	0.76	294,	0.34	$0.21 \sin(gt+163)$
1875-79:	153	„	167	„	352	„	37	„	$0.26 \sin(gt+141)$
1880-84:	320	„	165	„	210	„	139	„	$0.31 \sin(gt+162)$
1885-89:	126	„	162	„	68	„	242	„	$0.46 \sin(gt+139)$
1890-94:	293	„	160	„	286	„	344	„	$0.24 \sin(gt+249)$
1895-99:	100	„	158	„	144	„	86	„	$0.52 \sin(gt+121)$
1900-04:	266	„	156	„	2	„	190	„	$0.20 \sin(gt+182)$
1905-12:	123,	0.31	153	„	177,	0.46	215,	-0.13	$0.45 \sin(gt+156)$

The compound wave $\overline{k} \sin(gt + \overline{K})$ is found from the formula

$$\overline{k} \cos \overline{K} = \Sigma \, Rk \cos(K + \Delta K)$$
$$\overline{k} \sin \overline{K} = \Sigma \, Rk \sin(K + \Delta K)$$

and R and ΔK from the equ. (21).

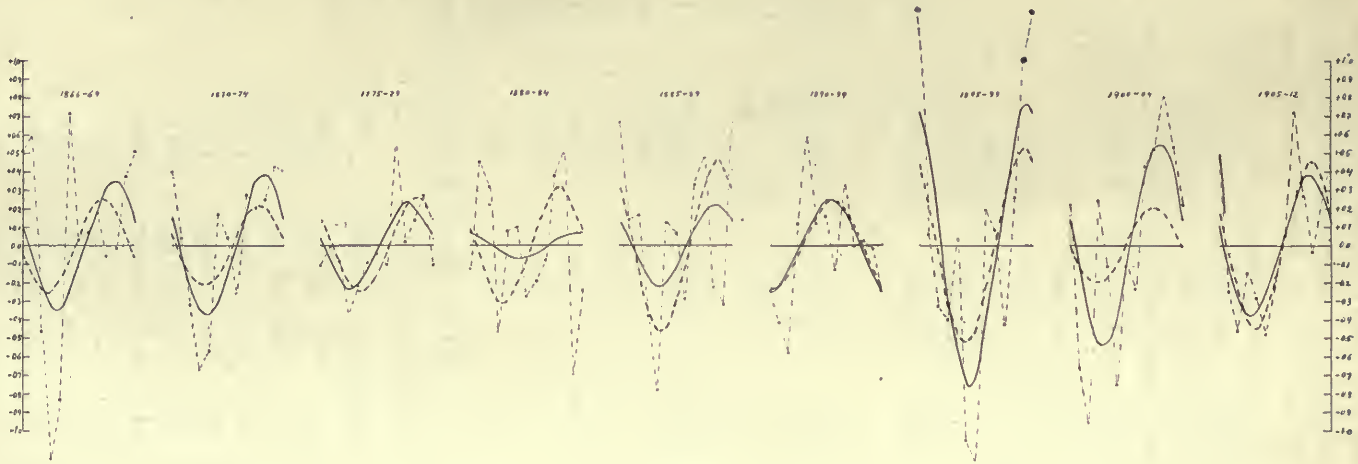

Fig. 4.

Uppsala temperature.

Comparison between observed variation in the period 29.600 days (————)
and the compound wave (— — — —) of the same period.
[Components: 29.380, 29.602, 29.792 and 29.947 days].

The observed sine-lines (Period 29.60) are obtained in the usual way and are based on the following variation of the 12 different means.

	$^{0}/_{12}$	$^{1}/_{12}$	$^{2}/_{12}$	$^{3}/_{12}$	$^{4}/_{12}$	$^{5}/_{12}$
1866—69:	+0.51	+0.59	—0.46	—1.15	—0.83	+0.72
70—74:	+ .40	+ .04	— .32	— .67	— .57	+ .17
75—79:	— .11	.00	— .08	— .38	— .22	— .09
80—84:	— .13	+ .45	+ .32	— .47	+ .08	+ 10
85—89:	+ .67	+ .14	+ .17	— .38	— .78	+ .13
90—94:	— .24	— .41	— .58	+ .12	+ .59	+ .12
95—99:	+1.25	+ .46	— .32	— .39	+ .05	—1.05
1900—04:	+0.22	— .66	— .96	+ .24	— .11	—0.75
05—12:	+ .49	— .24	— .46	— .15	— .28	— .48
1866-1912:	+.355	+.010	—.308	—.329	—.221	—.165

	$^{6}/_{12}$	$^{7}/_{12}$	$^{8}/_{12}$	$^{9}/_{12}$	$^{10}/_{12}$	$^{11}/_{12}$
1866—69:	+0.03	+0.08	+0.20	—0.05	—0.01	+0.38
70—74:	+ .04	— .26	+ .28	+ .21	+ .25	+ .43
75—79:	— .04	— .10	+ .56	+ .02	+ .14	+ .28
80—84:	— .27	— .19	— .08	+ .38	+ .50	— .70
85—89:	+ .07	— .29	+ .33	+ .48	— .22	— .31
90—94:	+ .16	— .12	+ .33	.00	+ .03	— .01
95—99:	—1.16	+ .20	+ .09	— .42	+ .27	+1.02
1900—04:	—0.11	— .23	+ .43	+ .52	+ .81	+0.53
05—12:	— .17	+ .05	+ .72	+ .27	— .03	+ .29
1866-1912:	—.164	—.090	+.345	+.168	+.183	+.216

From these residuals we obtain the following sine-lines, representing the observed variations. Fig. 4.

Period: 29.600 days.

1866—69: 0°.35 sin (gt + 160°)
70—74: 0.38 sin (gt + 155)
75—79: 0.24 sin (gt + 172)
80—84: 0.07 sin (gt + 110)
85—89: 0.22 sin (gt + 143)
90—94: 0.25 sin (gt + 260)
95 99: 0.76 sin (gt + 108)
1900—04: 0.54 sin (gt + 165)
05—12: 0.38 sin (gt + 164)
1866-1912: 0.29 sin (gt + 152)

We see at once that the phase-angles oscillate near the mean value 152°. Consequently the amplitude of the sine-curve analysed from the observations 1866—1912 has the same order of magnitude as the amplitudes corresponding to the observations from the differents lustrums. The phase of the sine-curve from the years 1890—94 do not agree with the mean value of the phase-angle, but it is only in this case, however, that the disturbing influence from neighbouring periods is great enough to cause a great change in the phase-angle (Fig. 4).

Compound wave.

Period: 36.464 days. Fig. 5.

	P = 36.187		36.464		Sum.
	K+ΔK,	R	K+ΔK,	R	P = 36.464
1880—84:	275°,	0.78	249°.	1.00	0°.53 sin (gt + 261°)
85—89:	53	„	„	„	0.10 sin (gt + 293)
90—94:	190	„	„	„	0.47 sin (gt + 223)
95—99:	328	„	„	„	0.42 sin (gt + 283)
1900—04:	105	„	„	„	0.18 sin (gt + 196)
1905—12:	285,	0.49	„	„	0.43 sin (gt + 261)

Observed variations. Periods 36.464 days.

	$^{0}/_{12}$	$^{1}/_{12}$	$^{2}/_{12}$	$^{3}/_{12}$	$^{4}/_{12}$	$^{5}/_{12}$
1880—84:	—0.72	—0.31	+0.10	—0.15	—0.75	+0.31
85—89:	— .93	+ .30	+ .56	+ .64	— .31	— .11
90—94:	— .82	— .79	— .54	— .14	— .49	+ .57
95—99:	— 1.11	— .12	— .16	+ .60	+ .93	+1.19
1900—04:	+ .35	— .62	— .57	— .08	— .21	— .14
05—12:	— .26	— .56	— .33	— .34	+ .09	— .12
1880—94:	—.826	—.265	+.041	+.119	—.514	+.265
1895-1912:	—.321	—.455	—.349	—.005	+.243	+.283
1880-1912:	—.550	—.369	—.172	+.052	—.101	+.275

	$^{6}/_{12}$	$^{7}/_{12}$	$^{8}/_{12}$	$^{9}/_{12}$	$^{10}/_{12}$	$^{11}/_{12}$
1880—84:	+0.29	—0.46	—0.07	+1.19	+0.46	+0.04
85—89:	+ .16	+ .07	+ .01	+ .11	+ .51	—1.06
90—94:	+ .48	+1.45	+ .70	+ .51	— .61	— .37
95—99:	— .03	— .04	— .42	— .51	— .17	— .09
1900—04:	+ .22	+ .59	— .44	+ .05	+ .31	+ .52
05—12:	+ .30	+ .25	+ .08	+ .49	— .02	+ .25
1880—94:	+.318	+.364	+.221	+.612	+.120	—.458
1895-1912:	+.189	+.263	—.200	+.092	+.030	+.231
1880-1912:	+.244	+.309	—.009	+.328	+.071	—.082

These residuals give the following sine-waves, corresponding to the period 36.464 days.

1880—84: $0^{\circ}.35 \sin(gt + 192^{\circ})$
85—89: $0.23 \sin(gt + 311)$
90—94: $0.87 \sin(gt + 244)$
95—99: $0.66 \sin(gt + 317)$
1900—04: $0.26 \sin(gt + 184)$
05—12: $0.35 \sin(gt + 218)$
1880—94: $0.40 \sin(gt + 241)$
1895-1912: $0.23 \sin(gt + 254)$
1880-1912: $0.31 \sin(gt + 247)$

As we se in fig. 5. the two terms of the periods 36.187 and 36.464 days give a better representation of the observed variation than the single term of the period 36.464 days. [The used values of the amplitudes and phase-angles, see list of terms.] From the observed variations in the years 1880—94 and 1895—1912 we can get a second approximation of the exact values of the constants. The difference of the phase-angles (254—241 = 13) corresponds to an error $\varDelta$P in the wavelength, determined by equation (26)

$$\varDelta P = \frac{P^2 (K'' - K')}{2\pi (T'' - T')} = \frac{(36.46)^2 \cdot 13}{360 \cdot 16.5 \cdot 365.25} = +0.008 \text{ days}$$

$$\text{or} \qquad P = 36.464 - 0.008 = 36.456 \text{ days.}$$

The corrected value of K is thus (Equ. 28)

$$254^{\circ} - 10^{\circ} = 244^{\circ}.$$

The most probable value of this term is thus

$$0^{\circ}.31 \sin\left(\frac{2\pi}{36.456} t + 244^{\circ}\right).$$

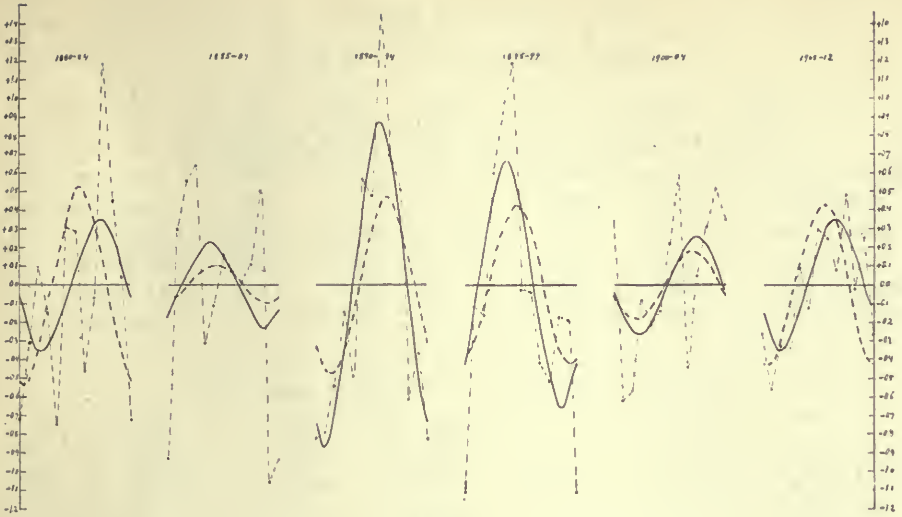

Fig 5.

Uppsala temperature.

Comparison between observed variation (————) in the period 36.464 days, and the compound wave — — — —) of the same period. [Components: 36.187 and 36.464 days.]

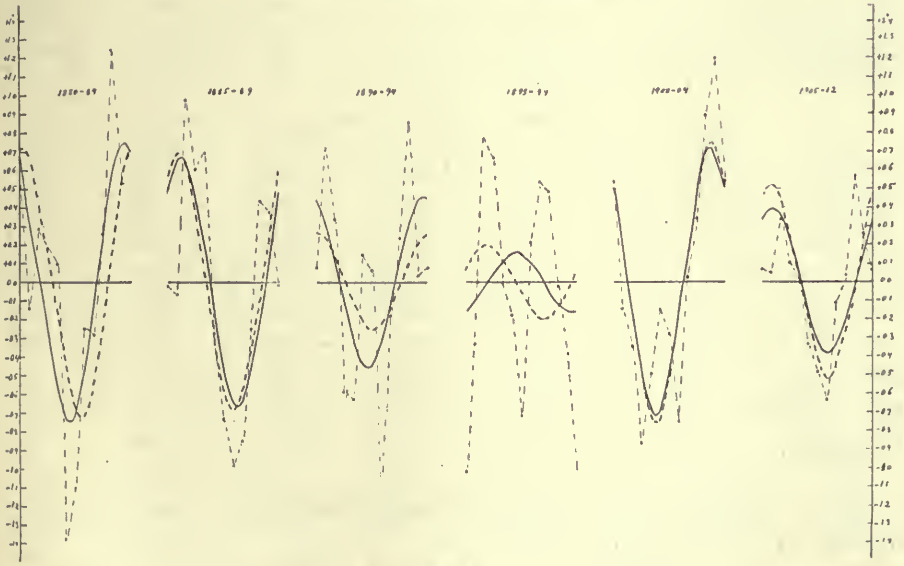

Fig. 6.

Uppsala temperature.

Comparison between observed variation (————) in the period 40.123 days, and the compound wave (— — — —) of the same period. [Components: 38.917, 39.417, 39.704, 40.123, 40.346 and 40.527 days.]

Compound wave.

Period: 40.123 days. Fig. 6.

P =	38.917	39.417	39.704	40.123	40.340	40.527	Sum.
	K+ΔK, R	K+ΔK, R	K+ΔK, R	K+ΔK, R	K+ΔK, R	K+ΔK, R	P = 40.123
1880-84:	100, -0.22	155, 0.22	222, 0.66	87, 1.00	141, 0.90	103, 0.69	0.72 sin (gt+119)
85-89:	248 ,,	88 ,,	35 ,,	,, ,,	49 ,,	299 ,,	0.69 sin (gt+ 56)
90-94:	36 ,,	22 ,,	209 ,,	,, ,,	317 ,,	135 ,,	0.26 sin (gt+ 85)
95-99:	184 ,,	315 ,,	22 ,,	,, ,,	225 ,,	331 ,,	0.20 sin (gt+ 17)
1900-04:	332 ,,	248 ,,	196 ,,	,, ,,	133 ,,	167 ,,	0.75 sin (gt+134)
1905-12:	272, 0.10	270, -0.19	61, 0.28	,, ,,	130, 0.75	314 0.34	0.52 sin (gt+ 52)

Observed variation. Period 40.123 days.

	0/12	1/12	2/12	3/12	4/12	5/12
1880—84:	+0.52	−0.14	+0.29	+0.18	+0.10	−1.38
85—89:	− .02	− .06	+ .98	+ .61	+ .70	− .27
90—94:	+ .08	+ .73	+ .34	− .59	− .63	+ .15
95—99:	−1.01	− .32	+ .78	+ .67	.00	− .17
1900—04:	+ .54	− .14	− .34	− .86	− .49	− .15
05—12:	+ .23	+ .16	+ .10	+ .46	+ .26	− .22
1880—94:	+ .19	+ .18	+ .54	+ .04	+ .06	− .50
1895-1912:	− .03	− .06	+ .17	+ .15	− .02	− .19
1880-1912:	+ .07	+ .05	+ .34	+ .10	+ .02	− .33

	6/12	7/12	8/12	9/12	10/12	11/12
1880—84:	−1.10	−0.25	−0.28	+0.26	+1.25	+0.53
85—99:	− .73	− .99	− .85	− .25	+ .44	+ .38
90—94:	+ .07	−1.07	− .18	+ .24	+ .86	+ .04
95—99:	− .71	+ .22	+ .54	+ .49	− .02	− .48
1900—04:	− .28	− .75	+ .03	+ .34	+ .90	+1.20
05—12:	− .27	− .81	.00	− .56	+ .21	+ .45
1880—94:	− .58	− .78	− .44	+ .08	+ .85	+ .32
1895-1912:	− .40	− .51	+ .16	− .01	+ .33	+ .40
1880-1912:	− .48	− .63	− .11	+ .03	+ .57	+ .36

These residuals give the following sine-waves nearly corresponding to the compound waves.

1880—84: 0°.75 sin (gt + 109°)
85—89: 0.67 sin (gt + 46)
90—94: 0.46 sin (gt + 104)
95—99: 0.16 sin (gt + 288)

1900—04: $0^{\circ}.72 \sin (gt + 136^{\circ})$
05—12: $0.39 \sin (gt + 57)$
1880—94: $0.53 \sin (gt + 86)$
1895-1912: $0.24 \sin (gt + 98)$
1880-1912: $0.37 \sin (gt + 90)$

From the two sine-curves of the years 1880—94 and 1895—1912 we obtain the following corrected term

$$0^{\circ}.38 \sin \left(\frac{2\pi}{40.114} t + 95^{\circ}\right).$$

This is the greatest term of a period between 20 and 40.5 days.

We could of course by smoothing out the residuals (or the original observations) considerably reduce the great deviation between the resulting sine-curve and the observed deviation in a certain period. The influence of eventual terms of periods nearly equal to the half, the third part, a. s. o. of the original period can however most easily be eliminated by calculating the variation that represents these fractional periods.

If for instance at the observations from 1880—1884 of the variation in the period 40.123 days we would eliminate the variation corresponding to $\frac{1}{2}P$, $\frac{1}{3}P$, $\frac{1}{4}P$, and $\frac{1}{6}P$, we should have the following table.

$P = 40.123$ days. 1880—84.

	$^{0}/_{12}$	$^{1}/_{12}$	$^{2}/_{12}$	$^{3}/_{12}$	$^{4}/_{12}$	$^{5}/_{12}$	$^{6}/_{12}$	$^{7}/_{12}$	$^{8}/_{12}$	$^{9}/_{12}$	$^{10}/_{12}$	$^{11}/_{12}$
	+0.52	—0.14	+0.29	+0.18	+0.10	—1.38	—1.10	—0.25	—0.28	+0.26	+1.25	+0.53
$(\frac{1}{2}P + \frac{1}{4}P + \frac{1}{6}P)$	— .29	— .20	.00	+ .22	+ .68	— .42	— .29	— .20	.00	+ .22	+ .68	— .42
$(\frac{1}{3}P + \frac{1}{6}P)$	+ .11	— .42	+ .14	+ .15	+ .11	— .42	+ .14	+ .15	+ .11	— .42	+ .14	+ .15
$(\frac{1}{6}P)$	+ .13	— .13	+ .13	— .13	+ .13	— .13	+ .13	— .13	+ .13	— .13	+ .13	— .13
(P)	+ .83	+ .35	+ .28	— .32	— .56	— .67	— .82	— .33	— .26	+ .33	+ .56	+ .67

The second line gives the means $\frac{{}^0/_{12} + {}^6/_{12}}{2}$, $\frac{{}^1/_{12} + {}^7/_{12}}{2}$ etc., i. e. the variation that repeats twice, or $\frac{1}{2}P + \frac{1}{4}P + \frac{1}{6}P$; the third line gives the means $\frac{{}^0/_{12} + {}^4/_{12} + {}^8/_{12}}{3}$, $\frac{{}^1/_{12} + {}^5/_{12} + {}^9/_{12}}{3}$ etc., i. e. the variation that repeats three times, or $\frac{1}{3}P + \frac{1}{6}P$. As $\frac{1}{6}P$ is taken twice we must separately compute this variation:

$$\left(\frac{{}^0/_{12} + {}^2/_{12} + {}^4/_{12} + {}^6/_{12} + {}^8/_{12} + {}^{10}/_{12}}{6}, \quad \frac{{}^1/_{12} + {}^3/_{12} + {}^5/_{12} + {}^7/_{12} + {}^9/_{12} + {}^{11}/_{12}}{6}\right).$$

The total correction, which must be added to the first line, is thus:

$$-\left(\frac{1}{2}P + \frac{1}{4}P + \frac{1}{6}P\right) - \left(\frac{1}{3}P + \frac{1}{6}P\right) + \left(\frac{1}{6}P\right).$$

The result is written in the last line. The variation in the period P found in this way gives with very small errors the sine-curve itself, and we can graphically find the amplitudes and the phases. (Fig. 3.) The above given sine-waves are nevertheless calculated trigonometrically.

If there is a greater variation corresponding to the period $\frac{1}{5}P$ or $\frac{1}{7}P$, this variation is of course not eliminated in this way.

We have seen that it is probable that at least the greater terms exist as an average for 5 years, but it is of interest to prove the existence for each one of the different years. For this purpose I have made a trial with the arbitrary period 28.50 days, and calculated the compound waves for the years 1890—1909 and compared them with the observed variations corresponding to the same period. I have then found the following results.

	Compound wave calculated as a sum of terms of the periods 26.76, 26.92, 26.97, 27,10, 27.17, 27.30, 27.49, 27.71, 28.02, 28.13, 28.41, 28.50, 28.58, 28.82, 28.93, 29.38, 29.60, 29.79, 29.95 and 30.22 days. with reduced amplitudes.		Observed sine-waves corresponding to the period 28.50 days.	
1890	$0^0.60 \sin\left(\frac{2\pi}{28.50}t + 61^0\right)$		$0^0.77 \sin\left(\frac{2\pi}{28.50}t + 30^0\right)$	
1891	0.75	210	1.00	193
1892	0.40	204	0.68	197
1893	0.17	9	0.50	278

1894	0°.54	64°	0°.70	80°
1895	0.24	63	0.08	315
1896	0.16	64	0.26	312
1897	1.07	186	1.38	180
1898	0.39	66	0.72	10
1899	0.64	131	1.24	120
1900	0.58	20	0.58	12
1901	0.38	17	0.38	30
1902	0.56	218	0.30	210
1903	0.52	223	0.98	210
1904	0.73	146	0.60	152
1905	0.34	233	0.40	192
1906	1.06	297	1.08	325
1907	0.30	293	0.10	305
1908	0.49	209	0.38	180
1909	0.56	276	0.50	295

Table of the reduced amplitudes.

P =	26.76	26.92	26.97	27.10	27.17	27.30	27.49	27.71	28.02	28.13
Rk =	0.04	0.05	0.05	0.10	0.09	0.11	0.13	0.21	0.19	0.26
P =	28.41	28.50	28.58	28.82	28.93	29.38	29.60	29.79	29.95	30.22
Rk =	0.17	0.19	0.18	0.16	0.22	0.18	0.18	0.12	0.11	0.06

We see then that the greatest differences of phases occur in the years 1893, 1895 and 1896, but in all these cases the amplitude of one of the sine-waves is very small, the phase-angles then being bad determined. In all other cases there is a fairly good agreement.

Synthesis.

The calculation of compound waves is a kind of partial synthesis limited to a few elementary waves. If we add all terms we get a complete synthesis, which can be compared with the original observations. The made analysis is however not complete, as the examined wave-lengths are limited to the interval 20—40.5 days.

We could of course by taking into account a much greater number of terms of small amplitudes get a very good representation of the nature-curve, but the existence of these terms, cannot be proved without analysing a much greater series of observations in order to decide whether they are real (constant) terms on only formal ones.

In the following table I have made a synthesis of the terms in the table 2 for the year 1897 and compared the computed temperature-variations with the observed variations, represented by a smooth curve of 10 days' means for each day.

Uppsala Temperature.

Observed and Calculated Temperature-Deviations for the Year 1897. Fig. 7.

1897		Observed temperature[1]	Normal temperature	Deviation from normal temperature	10 days' means of temperature-deviations	Calculated temperature-deviations
Jan.	1	+ 2.0	— 5.2	+ 7.2	—	+ 1.62
	2	— 1.3	— 5.2	+ 3.9	—	
	3	— 4.6	— 5.3	+ 0.7	—	+ 0.36
	4	— 2.1	— 5.3	+ 3.2	—	
	5	— 1.6	— 5.4	+ 3.8	+ 2.45	— 1.56
	6	— 3.0	— 5.4	+ 2.4	+ 1.76	
	7	— 3.8	— 5.4	+ 1.6	+ 1.02	— 1.94
	8	— 5.4	— 5.5	+ 0.1	+ 1.15	
	9	— 5.4	— 5.5	+ 0.1	+ 0.95	— 2.31
	10	— 4.0	— 5.5	+ 1.5	+ 0.75	
	11	— 5.3	— 5.6	+ 0.3	+ 0.87	— 2.13
	12	— 9.1	— 5.6	— 3.5	+ 1.02	
	13	— 3.6	— 5.6	+ 2.0	+ 0.92	— 1.40
	14	— 4.4	— 5.6	+ 1.2	+ 0.82	
	15	— 3.9	— 5.7	+ 1.8	+ 0.70	+ 0.09
	16	— 2.1	— 5.7	+ 3.6	+ 0.23	
	17	— 2.6	— 5.7	+ 3.1	+ 0.11	+ 0.33
	18	— 6.6	— 5.7	— 0.9	— 0.36	
	19	— 6.7	— 5.7	— 1.0	— 1.01	+ 0.95
	20	— 5.3	— 5.7	— 0.4	— 1.54	
	21	— 10.2	— 5.8	— 4.4	— 2.66	+ 1.33
	22	— 10.5	— 5.8	— 4.7	— 3.70	
	23	— 8.5	— 5.8	— 2.7	— 3.43	+ 1.32
	24	— 11.1	— 5.8	— 5.3	— 3.32	
	25	— 9.2	— 5.7	— 3.5	— 3.30	+ 0.68
	26	— 13.3	— 5.7	— 7.6	— 3.61	
	27	— 13.0	— 5.7	— 7.3	— 4.39	+ 0.24
	28	— 3.9	— 5.7	+ 1.8	— 4.25	
	29	— 5.6	— 5.7	+ 0.1	— 5.01	— 0.37
	30	— 5.0	— 5.6	+ 0.6	— 5.94	
	31	— 13.1	— 5.6	— 7.5	— 6.47	— 0.16

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[1] Mean of observations for each hour.

1897		Observed temperature	Normal temperature	Deviation from normal temperature	10 days' means of temperature-deviations	Calculated temperature-deviations
Febr.	1	— 18.1	— 5.6	— 12.5	— 7.01	
	2	— 6.9	— 5.6	— 1.3	— 8.36	+ 0.20
	3	— 18.5	— 5.6	— 12.9	— 8.70	
	4	— 18.3	— 5.5	— 12.8	— 8.71	+ 0.42
	5	— 18.4	— 5.5	— 12.9	— 7.68	
	6	— 18.1	— 5.4	— 12.7	— 6.38	+ 0.53
	7	— 17.1	— 5.4	— 11.7	— 6.21	
	8	— 8.6	— 5.3	— 3.3	— 4.95	+ 0.55
	9	— 4.8	— 5.3	+ 0.5	— 4.41	
	10	— 2.4	— 5.2	+ 2.8	— 4.39	+ 0.23
	11	— 4.7	— 5.2	+ 0.5	— 3.26	
	12	— 4.7	— 5.1	+ 0.4	— 1.15	— 0.71
	13	— 5.4	— 5.1	— 0.3	— 0.15	
	14	— 12.4	— 5.0	— 7.4	+ 0.51	— 1.43
	15	— 17.6	— 4.9	— 12.7	+ 0.97	
	16	— 6.3	— 4.9	— 1.4	+ 1.56	— 2.03
	17	+ 4.6	— 4.8	+ 9.4	+ 2.01	
	18	+ 1.9	— 4.8	+ 6.7	+ 2.41	— 1.58
	19	+ 2.4	— 4.7	+ 7.1	+ 3.85	
	20	+ 2.8	— 4.6	+ 7.4	+ 5.65	— 1.41
	21	+ 1.9	— 4.5	+ 6.4	+ 6.65	
	22	+ 0.5	— 4.4	+ 4.9	+ 6.34	— 0.30
	23	— 0.6	— 4.3	+ 3.7	+ 6.04	
	24	+ 2.8	— 4.2	+ 7.0	+ 5.73	+ 1.21
	25	+ 1.2	— 4.1	+ 5.3	+ 5.42	
	26	+ 4.6	— 4.0	+ 8.6	+ 5.20	+ 2.56
	27	+ 2.4	— 3.9	+ 6.3	+ 5.03	
	28	— 0.1	— 3.8	+ 3.7	+ 5.22	+ 3.22
March	1	+ 0,3	— 3.7	+ 4.0	+ 4.85	
	2	+ 0.7	— 3.6	+ 4.3	+ 4.50	+ 2.89
	3	+ 0.7	— 3.5	+ 4.2	+ 3.58	
	4	— 0.2	— 3.4	+ 3.2	+ 2.89	+ 1.92
	5	+ 2.3	— 3.3	+ 5.6	+ 2.28	
	6	+ 0.1	— 3.2	+ 3.3	+ 1.65	+ 0.32
	7	— 1.3	— 3.1	+ 1.8	+ 0.96	
	8	— 3.5	— 2.9	— 0.6	+ 0.16	— 1.56
	9	— 3.4	— 2.8	— 0.6	— 0.46	

1897		Observed temperature	Normal temperature	Deviation from normal temperature	10 days' means of temperature-deviations	Calculated temperature-deviations
March	10	— 5.1	— 2.7	— 2.4	— 1.36	— 2.98
	11	— 4.8	— 2.5	— 2.3	— 2.09	
	12	— 5.0	— 2.4	— 2.6	— 2.28	— 3.66
	13	— 6.1	— 2.3	— 3.8	— 2.04	
	14	— 5.1	— 2.1	— 3.0	— 1.91	— 3.49
	15	— 5.4	— 2.0	— 3.4	— 1.86	
	16	— 5.8	— 1.8	— 4.0	— 2.23	— 2.28
	17	— 1.7	— 1.6	— 0.1	— 2.72	
	18	+ 0.3	— 1.5	+ 1.8	— 2.68	— 0.87
	19	— 0.6	— 1.3	+ 0.7	— 2.31	
	20	— 3.1	— 1.2	— 1.9	— 1.95	+ 0.92
	21	— 7.1	— 1.1	— 6.0	— 1.66	
	22	— 8.4	— 0.9	— 7.5	— 1.76	+ 2.67
	23	— 4.2	— 0.8	— 3.4	— 2.06	
	24	+ 0.1	— 0.6	+ 0.7	— 2.10	+ 3.46
	25	— 0.2	— 0.4	+ 0.2	— 1.77	
	26	— 1.4	— 0.3	— 1.1	— 1.23	+ 3.43
	27	— 1.2	— 0.1	— 1.1	— 0.42	
	28	— 1.2	0.0	— 1.2	— 0.17	+ 2.55
	29	+ 0.5	+ 0.2	+ 0.3	— 0.45	
	30	+ 1.8	+ 0.4	+ 1.4	— 0.70	+ 1.15
	31	0.0	+ 0.6	— 0.6	— 0.86	
April	1	+ 1.4	+ 0.8	+ 0.6	— 0.88	— 0.44
	2	+ 0.1	+ 1.0	— 0.9	— 0.89	
	3	— 0.9	+ 1.2	— 2.1	— 0.99	— 1.55
	4	— 0.9	+ 1.4	— 2.3	— 1.20	
	5	— 1.1	+ 1.6	— 2.7	— 1.14	— 2.46
	6	+ 0.5	+ 1.8	— 1.3	— 1.16	
	7	+ 0.7	+ 2.0	— 1.3	— 1.05	— 2.28
	8	+ 1.5	+ 2.2	— 0.7	— 0.83	
	9	+ 1.6	+ 2.3	— 0.7	— 0.14	— 1.50
	10	+ 2.5	+ 2.5	0.0	+ 0.40	
	11	+ 3.2	+ 2.8	+ 0.4	+ 0.60	— 0.84
	12	+ 3.2	+ 3.0	+ 0.2	+ 0.88	
	13	+ 3.3	+ 3.2	+ 0.1	+ 0.79	— 0.52
	14	+ 8.1	+ 3.5	+ 4.6	+ 0.49	
	15	+ 6.5	+ 3.8	+ 2.7	+ 0.24	— 0.37

1897		Observed temperature	Normal temperature	Deviation from normal temperature	10 days' means of temperature-deviations	Calculated temperature-deviations
April	16	+ 4.7	+ 4.0	+ 0.7	— 0.14	
	17	+ 5.6	+ 4.1	+ 1.5	— 0.36	+ 0.61
	18	+ 2.7	+ 4.3	— 1.6	— 0.45	
	19	+ 0.8	+ 4.5	— 3.7	— 0.90	+ 0.68
	20	+ 2.2	+ 4.7	— 2.5	— 0.98	
	21	+ 1.5	+ 4.9	— 3.4	— 0.68	+ 0.51
	22	+ 3.1	+ 5.1	— 2.0	— 0.35	
	23	+ 4.5	+ 5.3	— 0.8	— 0.39	+ 0.47
	24	+ 5.6	+ 5.5	+ 0.1	+ 1.29	
	25	+ 7.6	+ 5.7	+ 1.9	+ 2.02	+ 0.27
	26	+ 9.6	+ 5.9	+ 3.7	+ 2.73	
	27	+ 10.9	+ 6.1	+ 4.8	+ 3.22	+ 0.39
	28	+ 12.1	+ 6.3	+ 5.8	+ 3.63	
	29	+ 11.8	+ 6.5	+ 5.3	+ 3.95	+ 0.88
	30	+ 11.5	+ 6.7	+ 4.8	+ 4.07	
May	1	+ 10.6	+ 6.9	+ 3.7	+ 3.90	+ 1.09
	2	+ 9.9	+ 7.0	+ 2.9	+ 3.47	
	3	+ 10.5	+ 7.2	+ 3.3	+ 2.92	+ 0.61
	4	+ 10.7	+ 7.4	+ 3.3	+ 2.36	
	5	+ 10.7	+ 7.6	+ 3.1	+ 1.99	+ 0.48
	6	+ 9.8	+ 7.8	+ 2.0	+ 1.44	
	7	+ 8.5	+ 8.0	+ 0.5	+ 1.03	+ 0.19
	8	+ 8.5	+ 8.2	+ 0.3	+ 0.29	
	9	+ 10.1	+ 8.4	+ 1.7	— 0.20	— 0.33
	10	+ 7.6	+ 8.5	— 0.9	— 0.59	
	11	+ 6.9	+ 8.7	— 1.8	— 0.56	— 1.01
	12	+ 7.7	+ 8.9	— 1.2	— 0.33	
	13	+ 5.0	+ 9.1	— 4.1	— 0.02	— 1.44
	14	+ 7.7	+ 9.3	— 1.6	— 0.28	
	15	+ 8.7	+ 9.5	— 0.8	— 0.26	— 1.58
	16	+ 12.0	+ 9.7	+ 2.3	+ 0.05	
	17	+ 12.6	+ 9.8	+ 2.8	+ 0.38	— 1.19
	18	+ 13.4	+ 10.0	+ 3.4	+ 0.73	
	19	+ 9.3	+ 10.2	— 0.9	+ 0.61	— 0.85
	20	+ 9.7	+ 10.4	— 0.7	+ 0.52	
	21	+ 11.9	+ 10.6	+ 1.3	+ 0.21	— 0.18
	22	+ 12.9	+ 10.8	+ 2.1	+ 0.07	

1897		Observed temperature	Normal temperature	Deviation from normal temperature	10 days' means of temperature-deviations	Calculated temperature-deviation
May	23	+ 10.4	+ 11.0	— 0.6	+ 0.05	+ 0.20
	24	+ 8.4	+ 11.2	— 2.8	+ 0.39	
	25	+ 9.7	+ 11.4	— 1.7	+ 0.94	+ 0.99
	26	+ 10.8	+ 11.6	— 0.8	+ 1.43	
	27	+ 13.2	+ 11.8	+ 1.4	+ 1.86	+ 1.47
	28	+ 15.2	+ 12.0	+ 3.2	+ 2.68	
	29	+ 14.7	+ 12.2	+ 2.5	+ 3.53	+ 1.48
	30	+ 17.2	+ 12.4	+ 4.8	+ 4.01	
	31	+ 18.8	+ 12.6	+ 6.2	+ 4.65	+ 1.54
June	1	+ 19.2	+ 12.8	+ 6.4	+ 4.15	
	2	+ 20.5	+ 12.9	+ 7.6	+ 3.18	+ 1.02
	3	+ 18.8	+ 13.1	+ 5.7	+ 2.33	
	4	+ 16.3	+ 13.2	+ 3.1	+ 1.45	+ 0.42
	5	+ 19.0	+ 13.4	+ 5.6	+ 0.66	
	6	+ 10.0	+ 13.6	— 3.6	— 0.21	— 0.35
	7	+ 7.2	+ 13.7	— 6.5	— 0.93	
	8	+ 7.8	+ 13.8	— 6.0	— 1.15	— 0.80
	9	+ 10.0	+ 14.0	— 4.0	— 0.88	
	10	+ 12.4	+ 14.1	— 1.7	— 1.05	— 1.12
	11	+ 11.9	+ 14.2	— 2.3	— 0.64	
	12	+ 14.7	+ 14.3	+ 0.4	— 0.19	— 1.59
	13	+ 18.0	+ 14.5	+ 3.5	+ 0.19	
	14	+ 20.4	+ 14.6	+ 5.8	+ 0.55	— 1.76
	15	+ 18.7	+ 14.8	+ 3.9	+ 0.79	
	16	+ 15.4	+ 14.9	+ 0.5	+ 1.09	— 1.67
	17	+ 13.0	+ 15.0	— 2.0	+ 1.35	
	18	+ 12.9	+ 15.1	— 2.2	+ 1.33	— 1.18
	19	+ 14.9	+ 15.3	— 0.4	+ 1.14	
	20	+ 16.1	+ 15.4	+ 0.7	+ 0.61	0.00
	21	+ 16.2	+ 15.5	+ 0.7	+ 0.40	
	22	+ 18.6	+ 15.6	+ 3.0	+ 0.44	+ 1.24
	23	+ 19.0	+ 15.7	+ 3.3	+ 0.70	
	24	+ 19.7	+ 15.8	+ 3.9	+ 1.09	+ 1.90
	25	+ 14.5	+ 15.9	— 1.4	+ 1.47	
	26	+ 14.4	+ 16.0	— 1.6	+ 1.53	+ 2.31
	27	+ 14.5	+ 16.1	— 1.6	+ 1.00	
	28	+ 16.6	+ 16.2	+ 0.4	+ 0.59	+ 2.19

1897		Observed temperature	Normal temperature	Deviation from normal temperature	10 days' means of temperature-deviations	Calculated temperature-deviation
June	29	+ 19.8	+ 16.3	+ 3.5	− 0.04	
	30	+ 20.8	+ 16.3	+ 4.5	− 0.16	+ 1.50
July	1	+ 17.7	+ 16.4	+ 1.3	− 0.15	
	2	+ 14.2	+ 16.5	− 2.3	− 0.31	+ 0.42
	3	+ 15.8	+ 16.6	− 0.8	− 0.68	
	4	+ 14.2	+ 16.6	− 2.4	− 1.19	− 0.68
	5	+ 14.1	+ 16.7	− 2.6	− 1.71	
	6	+ 15.2	+ 16.7	− 1.5	− 2.12	− 1.86
	7	+ 13.6	+ 16.8	− 3.2	− 1.98	
	8	+ 13.5	+ 16.8	− 3.3	− 1.82	− 2.35
	9	+ 15.3	+ 16.9	− 1.6	− 1.32	
	10	+ 16.2	+ 16.9	− 0.7	− 0.78	− 2.38
	11	+ 14.1	+ 16.9	− 2.8	− 0.37	
	12	+ 16.1	+ 17.0	− 0.9	+ 0.13	− 1.96
	13	+ 17.8	+ 17.0	+ 0.8	+ 0.84	
	14	+ 19.6	+ 17.0	+ 2.6	+ 1.15	− 0.87
	15	+ 19.8	+ 17.0	+ 2.8	+ 1.30	
	16	+ 19.6	+ 17.0	+ 2.6	+ 1.80	+ 0.51
	17	+ 18.8	+ 17.0	+ 1.8	+ 2.14	
	18	+ 20.8	+ 17.0	+ 3.8	+ 2.19	+ 1.46
	19	+ 18.5	+ 17.0	+ 1.5	+ 2.37	
	20	+ 17.8	+ 17.0	+ 0.8	+ 2.47	+ 2.12
	21	+ 19.2	+ 17.0	+ 2.2	+ 2.47	
	22	+ 19.4	+ 16.9	+ 2.5	+ 2.44	+ 2.01
	23	+ 18.2	+ 16.9	+ 1.3	+ 2.24	
	24	+ 21.3	+ 16.9	+ 4.4	+ 2.39	+ 1.57
	25	+ 20.7	+ 16.9	+ 3.8	+ 2.81	
	26	+ 19.5	+ 16.9	+ 2.6	+ 2.85	+ 0.87
	27	+ 18.3	+ 16.8	+ 1.5	+ 2.85	
	28	+ 18.6	+ 16.8	+ 1.8	+ 2.82	+ 0.12
	29	+ 19.8	+ 16.8	+ 3.0	+ 2.33	
	30	+ 21.7	+ 16.7	+ 5.0	+ 2.06	− 0.80
	31	+ 19.3	+ 16.7	+ 2.6	+ 1.92	
Aug.	1	+ 19.1	+ 16.6	+ 2.5	+ 2.08	− 1.15
	2	+ 17.6	+ 16.6	+ 1.0	+ 2.37	
	3	+ 16.0	+ 16.5	− 0.5	+ 2.48	− 1.36
	4	+ 17.6	+ 16.5	+ 1.1	+ 2.36	

1897		Observed temperature	Normal temperature	Deviation from normal temperature	10 days' means of temperature-deviations	Calculated temperature-deviations
Aug.	5	+ 17.6	+ 16.4	+ 1.2	+ 2.53	— 1.16
	6	+ 19.4	+ 16.3	+ 3.1	+ 2.50	
	7	+ 21.0	+ 16.3	+ 4.7	+ 2.45	— 0.95
	8	+ 20.3	+ 16.2	+ 4.1	+ 2.60	
	9	+ 19.9	+ 16.1	+ 3.8	+ 2.54	— 0.36
	10	+ 20.3	+ 16.0	+ 4.3	+ 2.64	
	11	+ 18.1	+ 15.9	+ 2.2	+ 2.69	+ 0.10
	12	+ 16.3	+ 15.8	+ 0.5	+ 2.35	
	13	+ 16.7	+ 15.7	+ 1.0	+ 2.13	+ 0.58
	14	+ 16.1	+ 15.6	+ 0.5	+ 1.92	
	15	+ 17.8	+ 15.6	+ 2.2	+ 1.63	+ 0.40
	16	+ 19.1	+ 15.5	+ 3.6	+ 1.47	
	17	+ 16.7	+ 15.4	+ 1.3	+ 1.51	+ 0.42
	18	+ 17.2	+ 15.3	+ 1.9	+ 1.37	
	19	+ 16.9	+ 15.2	+ 1.7	+ 1.36	+ 0.50
	20	+ 16.5	+ 15.1	+ 1.4	+ 1.26	
	21	+ 15.6	+ 15.0	+ 0.6	+ 1.04	+ 0.75
	22	+ 15.7	+ 14.8	+ 0.9	+ 0.98	
	23	+ 14.3	+ 14.7	— 0.4	+ 0.92	+ 0.77
	24	+ 15.0	+ 14.6	+ 0.4	+ 0.84	
	25	+ 15.7	+ 14.5	+ 1.2	+ 0.80	+ 0.90
	26	+ 15.7	+ 14.3	+ 1.4	+ 0.81	
	27	+ 14.9	+ 14.2	+ 0.7	+ 0.81	+ 0.70
	28	+ 15.4	+ 14.1	+ 1.3	+ 0.93	
	29	+ 14.8	+ 13.9	+ 0.9	+ 1.01	+ 0.50
	30	+ 14.8	+ 13.8	+ 1.0	+ 1.01	
	31	+ 14.3	+ 13.6	+ 0.7	+ 0.62	+ 0.17
Sept.	1	+ 14.4	+ 13.5	+ 0.9	+ 0.07	
	2	+ 14.2	+ 13.4	+ 0.8	— 0.57	— 0.61
	3	+ 14.4	+ 13.2	+ 1.2	— 1.01	
	4	+ 14.2	+ 13.0	+ 1.2	— 1.59	— 1.31
	5	+ 10.3	+ 12.8	— 2.5	— 1.99	
	6	+ 7.4	+ 12.7	— 4.8	— 2.36	— 1.49
	7	+ 7.5	+ 12.6	— 5.1	— 2.42	
	8	+ 8.9	+ 12.4	— 3.5	— 2.54	— 1.83
	9	+ 7.5	+ 12.3	— 4.8	— 2.76	
	10	+ 8.8	+ 12.1	— 3.3	— 2.44	— 1.61

1897		Observed temperature	Normal temperature	Deviation from normal temperature	10 days' means of temperature-deviations	Calculated temperature-deviation
Sept.	11	+ 9.2	+ 12.0	— 2.8	— 1.86	
	12	+ 12.0	+ 11.8	+ 0.2	— 1.13	— 1.25
	13	+ 11.6	+ 11.6	0.0	— 0.69	
	14	+ 10.4	+ 11.4	— 1.0	— 0.31	— 0.53
	15	+ 11.9	+ 11.2	+ 0.7	+ 0.18	
	16	+ 12.0	+ 11.0	+ 1.0	+ 0.51	+ 0.65
	17	+ 13.0	+ 10.8	+ 2.2	+ 0.38	
	18	+ 11.5	+ 10.6	+ 0.9	+ 0.34	+ 1.96
	19	+ 9.5	+ 10.5	— 1.0	+ 0.65	
	20	+ 11.9	+ 10.3	+ 1.6	+ 0.96	+ 2.29
	21	+ 10.6	+ 10.1	+ 0.5	+ 0.98	
	22	+ 8.9	+ 10.0	— 1.1	+ 0.90	+ 2.70
	23	+ 9.4	+ 9.8	— 0.4	+ 0.67	
	24	+ 11.7	+ 9.6	+ 2.1	+ 0.42	+ 2.31
	25	+ 13.2	+ 9.4	+ 3.8	+ 0.11	
	26	+ 10.4	+ 9.2	+ 1.2	— 0.29	+ 1.28
	27	+ 10.4	+ 9.0	+ 1.4	— 0.52	
	28	+ 7.4	+ 8.8	— 1.4	— 1.00	+ 0.38
	29	+ 5.1	+ 8.6	— 3.5	— 1.85	
	30	+ 4.7	+ 8.4	— 3.7	— 2.89	— 0.60
Oct.	1	+ 6.9	+ 8.2	— 1.3	— 3.52	
	2	+ 4.6	+ 8.0	— 3.4	— 3.87	— 1.49
	3	+ 2.6	+ 7.8	— 5.2	— 4.07	
	4	+ 1.2	+ 7.6	— 6.4	— 3.61	— 2.61
	5	+ 0.9	+ 7.5	— 6.6	— 3.26	
	6	+ 2.2	+ 7.3	— 5.1	— 2.99	— 2.85
	7	+ 5.0	+ 7.1	— 2.1	— 2.55	
	8	+ 3.5	+ 6.9	— 3.4	— 2.09	— 2.44
	9	+ 7.8	+ 6.7	+ 1.1	— 1.65	
	10	+ 6.3	+ 6.5	— 0.2	— 1.14	— 1.99
	11	+ 7.7	+ 6.3	+ 1.4	— 0.35	
	12	+ 7.1	+ 6.1	+ 1.0	+ 0.40	— 0.94
	13	+ 5.3	+ 5.9	— 0.6	+ 1.41	
	14	+ 3.8	+ 5.8	— 2.0	+ 1.81	+ 0.46
	15	+ 4.1	+ 5.6	— 1.5	+ 2.06	
	16	+ 8.2	+ 5.4	+ 2.8	+ 2.03	+ 1.51
	17	+ 10.6	+ 5.2	+ 5.4	+ 2.34	

1897		Observed temperature	Normal temperature	Deviation from normal temperature	10 days' means of temperature-deviations	Calculated temperature-deviations
Oct.	18	+ 11.7	+ 5.0	+ 6.7	+ 2.82	+ 2.59
	19	+ 9.9	+ 4.8	+ 5.1	+ 3.21	
	20	+ 6.9	+ 4.6	+ 2.3	+ 3.30	+ 2.87
	21	+ 5.5	+ 4.4	+ 1.1	+ 2.93	
	22	+ 8.3	+ 4.2	+ 4.1	+ 2.12	+ 2.68
	23	+ 8.2	+ 4.0	+ 4.2	+ 1.19	
	24	+ 5.8	+ 3.9	+ 1.9	+ 0.61	+ 2.23
	25	+ 3.1	+ 3.7	— 0.6	+ 0.21	
	26	+ 2.6	+ 3.5	— 0.9	+ 0.23	+ 1.31
	27	+ 0.7	+ 3.4	— 2.7	— 0.26	
	28	+ 0.6	+ 3.2	— 2.6	— 0.84	— 0.02
	29	+ 2.3	+ 3.0	— 0.7	— 1.20	
	30	+ 1.1	+ 2.8	— 1.7	— 1.11	— 1.36
	31	+ 3.9	+ 2.6	+ 1.3	— 0.93	
Nov.	1	+ 1.6	+ 2.4	— 0.8	— 0.81	— 1.94
	2	+ 0.7	+ 2.3	— 1.6	— 0.77	
	3	+ 0.4	+ 2.1	— 1.7	— 0.69	— 3.45
	4	+ 2.2	+ 1.9	+ 0.3	— 0.57	
	5	+ 2.6	+ 1.7	+ 0.9	— 0.48	— 2.61
	6	+ 0.1	+ 1.6	— 1.5	— 0.24	
	7	— 0.8	+ 1.4	— 2.2	— 0.21	— 2.01
	8	+ 1.3	+ 1.2	+ 0.1	+ 0.61	
	9	+ 0.5	+ 1.0	— 0.5	+ 1.59	— 0.95
	10	+ 3.1	+ 0.9	+ 2.2	+ 1.97	
	11	+ 2.3	+ 0.7	+ 1.6	+ 1.72	+ 0.12
	12	— 0.7	+ 0.6	— 1.3	+ 1.39	
	13	+ 6.9	+ 0.4	+ 6.5	+ 1.52	+ 1.07
	14	+ 10.3	+ 0.2	+ 10.1	+ 2.04	
	15	+ 4.8	+ 0.1	+ 4.7	+ 2.35	+ 1.63
	16	+ 4.0	0.0	— 4.0	+ 2.63	
	17	— 5.7	— 0.2	— 5.5	+ 3.49	+ 1.78
	18	+ 1.1	— 0.3	+ 1.4	+ 3.10	
	19	+ 4.2	— 0.5	+ 4.7	+ 1.78	+ 1.17
	20	+ 4.7	— 0.6	+ 5.3	+ 0.39	
	21	+ 3.6	— 0.8	+ 4.4	+ 0.41	+ 0.74
	22	+ 6.3	— 1.0	+ 7.3	+ 0.94	
	23	+ 1.5	— 1.1	+ 2.6	+ 1.00	+ 0.14

1897		Observed temperature	Normal temperature	Deviation from normal temperature	10 days' means of temperature-deviations	Calculated temperature-deviations
Nov.	24	— 4.3	— 1.2	— 3.1	— 1.10	
	25	— 10.5	— 1.3	— 9.2	— 2.11	— 0.48
	26	— 5.3	— 1.5	— 3.8	— 2.89	
	27	— 1.9	— 1.7	— 0.2	— 3.41	— 0.68
	28	+ 0.2	— 1.8	+ 2.0	— 3.45	
	29	— 8.3	— 2.0	— 6.3	— 3.26	— 0.41
	30	— 16.9	— 2.1	— 14.8	— 2.60	
Dec.	1	— 5.6	— 2.2	— 3.4	— 2.24	+ 0.07
	2	— 0.3	— 2.4	+ 2.1	— 1.19	
	3	— 0.3	— 2.5	+ 2.2	— 1.70	+ 0.21
	4	— 3.8	— 2.6	— 1.2	— 0.67	
	5	— 5.3	— 2.7	— 2.6	+ 1.26	+ 0.41
	6	— 3.0	— 2.8	— 0.2	+ 1.89	
	7	+ 0.4	— 2.9	+ 3.3	+ 1.71	+ 0.58
	8	+ 0.9	— 3.0	+ 3.9	+ 1.69	
	9	+ 0.9	— 3.1	+ 4.0	+ 2.01	+ 0.32
	10	+ 1.3	— 3.2	+ 4.5	+ 2.73	
	11	— 0.4	— 3.3	+ 2.9	+ 3.31	+ 0.01
	12	— 3.1	— 3.4	+ 0.3	+ 3.76	
	13	— 1.6	— 3.6	+ 2.0	+ 4.12	— 0.98
	14	— 1.7	— 3.7	+ 2.0	+ 4.20	
	15	+ 0.8	— 3.8	+ 4.6	+ 3.91	— 0.95
	16	+ 1.7	— 3.9	+ 5.6	+ 3.83	
	17	+ 3.8	— 4.0	+ 7.8	+ 4.06	— 1.18
	18	+ 3.4	— 4.1	+ 7.5	+ 3.70	
	19	+ 0.6	— 4.2	+ 4.8	+ 2.84	— 0.82
	20	— 2.7	— 4.3	+ 1.6	+ 2.37	
	21	— 2.2	— 4.3	+ 2.1	+ 2.08	— 0.64
	22	— 1.8	— 4.4	+ 2.6	+ 1.95	
	23	— 6.1	— 4.5	— 1.6	+ 2.20	— 0.29
	24	— 11.2	— 4.6	— 6.6	+ 2.62	
	25	— 4.8	— 4.7	— 0.1	+ 3.52	+ 0.26
	26	— 2.0	— 4.7	+ 2.7	+ 4.04	
	27	+ 1.7	— 4.8	+ 6.5		+ 0.77
	28	+ 5.1	— 4.9	+ 10.0		
	29	+ 4.0	— 5.0	+ 9.0		+ 0.92
	30	+ 5.5	— 5.1	+ 10.6		
	31	+ 2.2	— 5.1	+ 7.3		+ 0.71

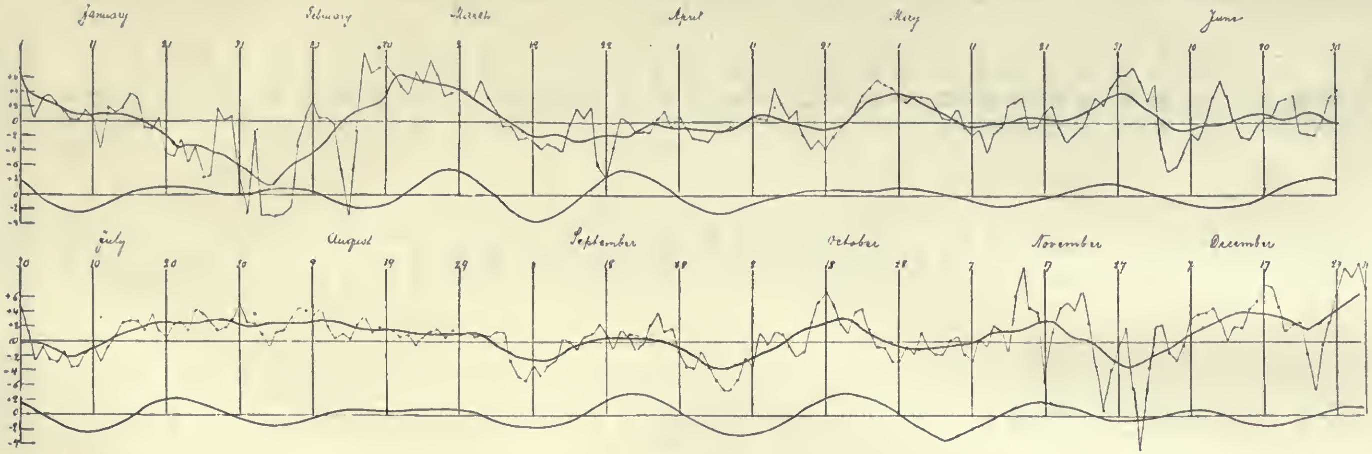

Fig. 7.

Uppsala Temperature 1897.

Comparison between observed 10 days' mean temperature (observed temp. — normal temp., the upper curve), and the sum of all terms of periods between 19.94 and 40.85 days (the lower curve).

The comparison between observed and calculated temperature for the year 1897 shows a good agreement during the time April 1 —Dec. 1. The great temperature-wave Jan. 17—March 10 has too long a period (about 50 days) to be reproduced by terms of periods between 20 and 40 days. The constant positive deviation July 12—Aug. 31 can neither be represented by terms of short periods.

The variations itselves of the observed temperature have of course a perceptible effect on the calculated temperature variations for the same time, as the observations from the year 1897 are used in the analysis. It is thus of interest to study this influence for a certain case. The temperature-wave Sept. 28—Okt. 28 has a total effect of $7^0.0$ C. and shows a good agreement with the calculated temperature-wave, which has an effect of $6^0.0$ (reduced to 10 days' mean: $5^0.0$). The effect of this single temperature-wave of a period equal to 30 days on the synthesis based on an analysis with 110 terms of periods between 20 and 40 days is certainly smaller than

$$\frac{110 \cdot 7.0}{400} = 1^0.92\ ^{1)}.$$

The analysed series contains 400 period equal to 30 days.

As the calculated temperature-wave reduced to 10 days' mean has a total variation of $5^0.0$, we see that this large amplitude principally arises from an analysis of variations in the temperature outside the said temperature-wave.

It is of course those temperature-waves, which have periods of about 30 days, which most exactly can be represented by the examined terms.

Periodogram.

I have appreciated the values of the different amplitudes corresponding to all the examined wave-lengths and made a graphical representation of these quantities (Fig. 8). We see in this periodogram that the different maxima of the amplitudes are well sepa-

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¹) If we divide the list of terms into 4 groups corresponding to the periods 20—25, 25—30, 30—35 and 35—40 days containing resp. 50, 30, 16 and 14 terms, and calculate the resp. factors of reduction for each group we will find a total effect of $1^0.56$.

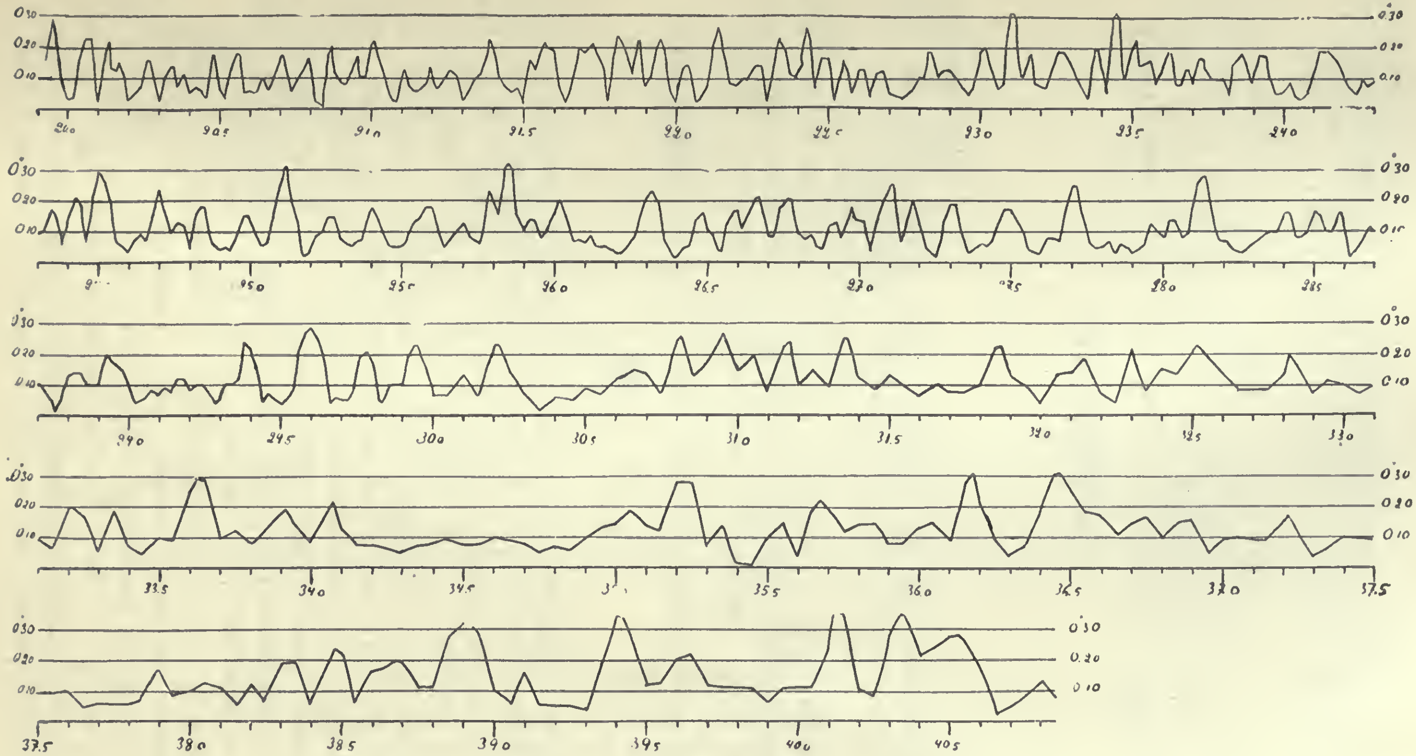

Fig. 8.

Periodogram.

Uppsala Temperature 1880—1912.

Amplitudes of sine-curves of different wave-lengths.

rated by small amplitudes and further, that if the maximal amplitudes represent constant terms there is a great number of them with amplitudes of the same order of magnitude.

In order to find the most probable amplitude I have made an investigation of the numbers of the amplitudes of different magnitude; the result is to be found in the following table.

Number of Amplitudes of different Magnitude.

	Wave-length: 19.04–30.00 days Interval: 0.02 days	Wave-length: 30.05—40.85 days Interval: 0.05 days
Amplitudes	Number of amplitudes	
0°.01, 0°.02	16	4
0.03, 0.04	46	9
0.05, 0.06	74	29
0.07, 0.08	68	35
0.09, 0.10	70	30
0.11, 0.12	30	23
0.13, 0.14	54	22
0.15, 0.16	40	16
0.17, 0.18	32	11
0.19, 0.20	40	12
0.21, 0.22	6	7
0.23, 0.24	12	4
0.25, 0.26	5	6
0.27, 0.28	5	8
0.29, 0.30	6	2
0.31, 0.32	2	2
0.33, 0.34	0	1
0.35, 0.36	0	1

The most probable amplitude is for each one of the two groups of wave-lengths equal to

0°.075.

Summary.

Assuming that a part of the temperature variations of Uppsala 1880—1912 can be expressed by a trigonometrical series of constant terms we have found the following facts.

I. No term of a period between 20 and 40.85 days has a greater amplitude than 0°.38 C. (Total effect 0°.76).

II. There is possibly a great number of terms of periods between 20 and 40 days, and about 50 terms with amplitudes greater than 0°.20 C.

III. To decide whether the terms are constant it is necessary to examine a much longer series of observations.

IV. The most accurate technic must be employed if an extrapolation of the temperature-curve will have any worth, as a small error in the wave-length gives rise to a large error in the extrapolated curve.

V. An investigation of the physical sense of the terms and of the relations between the constants can first be made when the terms are better determined and when similar analyses are made for different places and of kindred phenomena (barometric pressure, temperature of see-currents, etc.)

It is, I suppose, practical to study the records of the different seasons separately, as some of the terms possibly occur mainly in the winter and other in the summer. The number of terms will in this case be smaller and the amplitudes will be greater.

The analyses are most practical made with conveniently constructed analysators.

Note.

While working on this analysis, I found that similar analyses had been made by Prof. Arthur Schuster with regard to the periodicities of sunspots, Phil. Trans. of Roy. Soc., Ser. A, Vol. 206, and by Prof. H. H. Turner with regard to the Greenwich rainfall records, Quarterly Yournal of Roy. Meteor. Society, July 1911. By the said analyses Prof. Schuster on the one hand has established the period 11.12 years in the periodicities of the sunspots, and Prof. Turner on the other has discovered some periods in the rainfall records.

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